Proof that Assets Remain Bounded in Aygari's Model

Here I write a simpler proof of the result in Ayagari (1994) on behavior of cash in hand in the long run. Recall that the optimization problem for the household is given by

$$V(z) = \max u(z - a') + \beta \int V(Ra' + y)\, dF(y)$$

subject to

$$a' \geq \underline{a}$$

where $R = 1 + r$.

Note: I am going to omit the proof that $V(\cdot)$ is strictly concave, increasing and differentiable. These are standard and follow from SLP.

Claim 1. There exist $\underline{z}$ such that $\forall z > \underline{z}$, $a'(z) > \underline{a}$.

Proof. To prove this suppose that for a value z', $a'(z') = \underline{a}$ and $z'' < z'$ such that $a'(z'') > \underline{a}$. This means that at z'', we have

$$u'(z'' - a'(z'')) = \beta R \int V'(Ra'(z'') + y)\, dF(y)$$

and

$$u'(z' - \underline{a}) \geq \beta R \int V'(R\underline{a} + y)\, dF$$

We have

$$u'(z'' - a'(z'')) > u'(z' - \underline{a}) \geq \beta R \int V'(R\underline{a} + y)\, dF \geq \beta R \int V'(Ra'(z'') + y)\, dF$$

where all of the above follows from concavity of V and strict concavity of u. The above is a contradiction to the assumption that $a'(z') = \underline{a}$ and $a'(z'') > \underline{a}$. This means that if we consider the set $\mathcal{A} = \{z; a'(z) = \underline{a}\}$ it must be either a closed interval or the entire set $[R\underline{a} + y_{min}, \infty)$. If it is a closed interval then the claim is proved. If it is the entire set of possible values for z, then it must be that

$$V(z) = u(z - \underline{a}) + \beta \int V(R\underline{a} + y)\, dF$$

and as a result

$$\forall z, V'(z) = u'(z - \underline{a})$$

Moreover, since $a' \geq \underline{a}$ is binding, we must have that

$$u'(z - \underline{a}) \geq \beta R \int V'(R\underline{a} + y)\, dF = \beta R \int u'(r\underline{a} + y)\, dF$$

The LHS of the above inequality converges to 0 as z converges to ∞ while the right hand side is a positive number. This is a contradiction to the initial claim and thus we have established it. □

Claim 2. The policy function $a'(z)$ is increasing and strictly increasing when the borrowing constraint is slack. Moreover, as $z \to \infty$, $a'(z) \to \infty$.

Proof. As we have shown, the borrowing constraint binds for all values of $z \leq \underline{z}$ and slack for all higher values. Now, suppose consider $z_1 > z_2 > \underline{z}$ and let optimal assets be given by $a_1 \leq a_2$ respectively. We have

$$u'(z_1 - a_1) = \beta R \int V'(Ra_1 + y)\, dF$$
$$u'(z_2 - a_2) = \beta R \int V'(Ra_2 + y)\, dF$$

By the assumption, $z_1 > z_2$ and $a_1 \leq a_2$ which implies that $z_1 - a_1 > z_2 - a_2$. By concavity of V and u, we have

$$u'(z_1 - a_1) < u'(z_2 - a_2)$$
$$\int V'(Ra_1 + y)\, dF \geq \int V'(Ra_2 + y)\, dF$$

which is contradictory to the above Euler equations. This also implies that $c(z) = z - a'(z)$ is strictly increasing in z.

Evidently, if $\lim_{z\to\infty} a'(z) < \infty$, then it must be that $\lim_{z\to\infty} c(z) = \infty$. This implies that $u'(c(z)) \to 0$ while $\beta R \int V'(Ra'(z) + y)\, dF$ does not. This is a contradiction. The Euler equation combined with $a'(z) \to \infty$ implies that $c(z) \to \infty$. □

Claim 3. Consumption policy function satisfies ,$\forall \Delta > 0, 0 < c(z + \Delta) - c(z) < \Delta$.

Proof. The left inequality is already shown. In order to show the right inequality, notice that Euler equation at z is given by

$$u'(c(z)) = \beta R \int V'(Ra'(z) + y)\, dF$$

We therefore have

$$u'(c(z)+\Delta) < \beta R \int V'(Ra'(z)+y)\,dF$$

That is, at $a' = a'(z)$, a consumer with $z+\Delta$ cash-in-hand, would want to increase her asset holdings. Note that at $a' = a'(z)+\Delta$, we have

$$u'(c(z)) \geq \beta R \int V'(Ra'(z)+R\Delta+y)\,dF$$

which follows from concavity of V. This implies that $a'(z+\Delta) \in [a'(z), a'(z)+\Delta]$ since the objective is concave. This concludes the proof. □

Claim 4. There exists z^* such that for all $z \geq z^*$, $Ra'(z)+y_{max} \leq z$.

Proof. Let $z > \underline{z}$. Then the Euler equation is given by

$$u'(c(z)) = \beta R \mathbb{E}\left[u'(c(Ra'(z)+y))\right]$$

By Mean-value theorem, there must exist $c^*(y,z)$ between $c(Ra'(z)+y)$ and $c(\mathbb{E}[Ra'(z)+y])$ such that

$$u'(c(Ra'(z)+y)) - u'(c(Ra'(z)+y_{max})) = u''(c^*(y,z))\left(c(Ra'(z)+y) - c(Ra'(z)+y_{max})\right)$$

We can thus write the Euler equation – and simplify the notation, we have

$$u'(c_t) = \beta R\left\{u'(\bar{c}_{t+1}) + \mathbb{E}\left[u''\left(c^*_{t+1}\right)(c_{t+1}-\bar{c}_{t+1})\right]\right\}$$

where

$$\bar{c}_{t+1} = c(Ra'(z)+y_{max})$$

Note that this is not average consumption at $t+1$. We can write

$$\begin{aligned}
\left|\mathbb{E}\left[u''\left(c^*_{t+1}\right)(c_{t+1}-\bar{c}_{t+1})\right]\right| &\leq \left|\mathbb{E}\left[u''\left(c^*_{t+1}\right)\right]\right|(y_{max}-y_{min}) \\
&\leq \mathbb{E}\left[\left|u''\left(c^*_{t+1}\right)\right|\right](y_{max}-y_{min}) \\
&\leq \mathbb{E}\left[\frac{u'\left(c^*_{t+1}\right)}{c^*_{t+1}}\right]B(y_{max}-y_{min})
\end{aligned}$$

where in the above we have used Claim 3 and the fact that

$$-\frac{u''(c)\,c}{u'(c)} \leq B$$

The fact that the above holds, implies that

$$\frac{u'(c_2)}{u'(c_1)} \leq \left(\frac{c_2}{c_1}\right)^{-B}$$

where $c_2 > c_1$. This can be shown by writing the inequality as $-\frac{u''}{u'} \leq \frac{B}{c}$ and integrating both sides. Using the above inequality, we see that

$$\min\left\{1, \left(\frac{\overline{c}_{t+1}}{c^*_{t+1}}\right)^{-B}\right\} \leq \frac{u'\left(c^*_{t+1}\right)}{u'\left(\overline{c}_{t+1}\right)} \leq \max\left\{\left(\frac{c^*_{t+1}}{\overline{c}_{t+1}}\right)^{-B}, 1\right\}$$

Now by claim 3, we know that $\frac{\overline{c}_{t+1}}{c^*_{t+1}} \to 1$ as $z \to \infty$, since $c^*_{t+1} \to \infty$ as $z \to \infty$ which then implies that $\frac{u'\left(c^*_{t+1}\right)}{u'\left(\overline{c}_{t+1}\right)} \to 1$ as $z \to \infty$. Therefore, the above implies that

$$\begin{aligned}
\lim_{z\to\infty} \frac{\mathbb{E}u'(c_{t+1})}{u'(\overline{c}_{t+1})} &= 1 + \lim_{z\to\infty} \frac{\mathbb{E}\left[u''\left(c^*_{t+1}\right)\left(c_{t+1} - \overline{c}_{t+1}\right)\right]}{u'(\overline{c}_{t+1})} \\
&\leq 1 + \lim_{z\to\infty} \frac{\mathbb{E}\left[\frac{u'\left(c^*_{t+1}\right)}{c^*_{t+1}} B\left(y_{max} - y_{min}\right)\right]}{u'(\overline{c}_{t+1})} \\
&= 1 + B\left(y_{max} - y_{min}\right) \lim_{z\to\infty} \mathbb{E}\left[\frac{u'\left(c^*_{t+1}\right)}{\overline{c}_{t+1} u'\left(\overline{c}_{t+1}\right)}\right] \\
&= 1
\end{aligned}$$

This means that for any ε, there exists a z_ε high enough, so that for all higher z,

$$\frac{\mathbb{E}u'(c_{t+1})}{u'(\overline{c}_{t+1})} = \frac{u'(c_t)}{\beta R u'(\overline{c}_{t+1})} \leq 1 + \varepsilon$$

We can choose ε so that $(1+\varepsilon)\beta R < 1$. This means that $u'(c_t) = u'(c(z)) < u'(\overline{c}_{t+1}) = u'(c(Ra'(z) + y_{max}))$,$\forall z > z_\varepsilon$. Since u' is decreasing and $c(\cdot)$ is increasing, we have that $z > Ra'(z) + y_{max}$ which establishes the claim. $\square$

In the above proof, we used the fact that relative risk aversion is bounded above. Suppsoe that it is not and that utility function is given by

$$u(c) = -e^{-\psi c}$$

In this case, the solution of the unconstrained problem – without the borrowing constraints – can be found as follows:

$$-e^{-\gamma z+A} = \max_{a'} -e^{-\alpha(z-a')} - \beta \int e^{-\gamma(Ra'+y)+A} dF$$

where we have guessed that $V(z) = -e^{-\gamma z+A}$. The FOC is

$$e^{-\alpha(z-a')} = \beta R e^{-\gamma Ra'} \int e^{-\gamma(y+A)} dF$$

Taking logs, we have

$$-\alpha(z-a') = A + \log(\beta R) + \log\left(\mathbb{E}e^{-\gamma y}\right) - \gamma Ra' \rightarrow a' = \frac{\alpha}{\alpha+\gamma R}z + \kappa$$

Replacing in the definition of the value function, we get that

$$\gamma = \frac{\alpha\gamma R}{\alpha+\gamma R} \rightarrow \alpha + \gamma R = \alpha R \rightarrow \gamma = \frac{\alpha(R-1)}{R} = \frac{\alpha r}{1+r}$$

and

$$\begin{aligned} z' &= \frac{R\alpha}{\alpha+\gamma R}z + y + R\kappa \\ &= z + y + R\kappa \end{aligned}$$

We see that even though interest rates are low, cash-in-hand is a random walke with a drift and so it will not stay bounded. One can show that as cash-in-hand becomes large, the model with the borrowing constraint behaves like the one without. This proves that the assumption of bounded risk-aversion has some bite.