Optimal Rating Design

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Introduction

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  - Peer-to-peer platforms: eBay and Airbnb
  - Regulating insurance markets: Community ratings in health insurance exchanges under ACA
  - Certification of doctors and restaurants

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  - Peer-to-peer platforms: eBay and Airbnb
  - Regulating insurance markets: Community ratings in health insurance exchanges under ACA
  - Certification of doctors and restaurants

- Common feature:
  - Adverse selection and moral hazard
  - Intermediary observes information
  - Decides what to transmit to the other side

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- Key questions:
  - How should the intermediary transmit the information?
  - When is it optimal to hide some information?
  - When is it optimal to partition the state?

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- Try to answer them in a model with adverse selection and moral hazard

The Model

- Competitive Model of Adverse Selection and Moral Hazard

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- Unit continuum of buyers
  - Payoffs:

$$q - t$$

q: quality of the good purchased
t: transfer

The Model

- Unit continuum of sellers
  - Produce one vertically differentiated product
  - Choose quality q
  - Differ in cost of quality provision

$$\text{Cost} : C(q,\theta); \theta \sim F(\theta)$$

  - Payoffs

$$t - C(q,\theta)$$

The Model

Assumption. Cost function satisfies: $C_{qq} > 0, C_q > 0, C_\theta < 0, C_{\theta q} \leq 0$.

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Assumption. Cost function satisfies: $C_{qq} > 0, C_q > 0, C_\theta < 0, C_{\theta q} \leq 0$.

- First Best Efficient: maximize total surplus $q - C(q, \theta)$

$$C_q\left(q^{FB}(\theta), \theta\right) = 1$$

- Submodularity: $q^{FB}(\theta)$ is increasing in θ.
    - Higher θ's have lower marginal cost

Information Control

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- Buyers only see the signal by the intermediary

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  - S: set of signals
  - $\pi(\cdot|q) \in \Delta(S)$

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  - S: set of signals
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- Key statistic from the buyers perspective

$$\mathbb{E}[q|s]$$

Equilibrium

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- Sellers payoff

$$q(\theta) \in \arg\max_{q'} \int \hat{p}(s)\, \pi\left(ds|q'\right) - C\left(q', \theta\right) \tag{2}$$

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$$q(\theta) \in \arg\max_{q'} \int \hat{p}(s)\, \pi\left(ds|q'\right) - C\left(q', \theta\right) \tag{2}$$

Equilibrium: $\left(\{q(\theta)\}_{\theta\in\Theta}, u, \hat{p}(s)\right)$ that satisfy (1) and (2).

Interpretations

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- A bit of a stretch!: opaque financial markets
  - Regulators reveal information about prices

Rating Design Problem

- The goal: find (S, π) according to some objective
  - Pareto optimality of outcomes
  - Maximize revenue
  - Maximize information provided
  - etc.

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    - Problem is quite complicated and non-linear in (S, π)
- Second Step:
  - Characterize what Pareto optimal outcomes look like

Second Order Expectations

- The main complication: what matters for incentives is the second order expectations of the sellers

$$\mathbb{E}\left[\mathbb{E}\left[q|s\right]|q'\right]$$

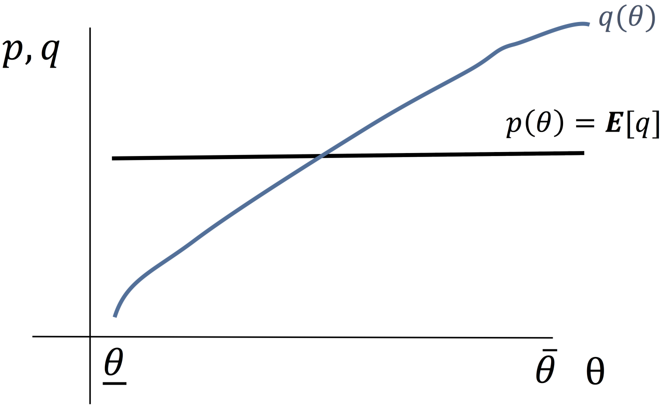
- Not so obvious how to characterize
- Our main result: provide a characterization for it

Definition. (Signaled qualities) For a quality profile $\{q(\theta)\}_{\theta\in\Theta}$ and arbitrary RS:

$$p(\theta) = \int_S \mathbb{E}\left[q|s\right]\pi\left(ds|q(\theta)\right)$$

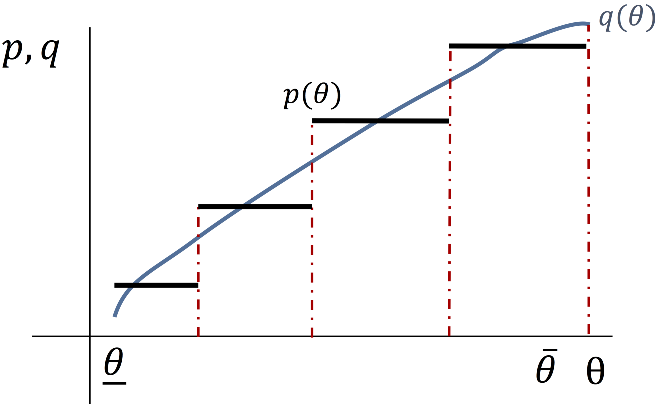
Examples

No information



Examples

Partition Information



Characterizing Rating Systems

- Let's start with discrete types $\Theta = \{\theta_1 < \cdots < \theta_N\}$ and distribution $F : \mathbf{f} = (f_1, \cdots, f_N)$
  - Boldface letters: vectors in $\mathbb{R}^N$
- Standard revelation-principle-type arugment leads to the following lemma

Lemma 1. If a vector of qualities, $\mathbf{q}$, and signaled qualities, $\mathbf{p}$ arise from an equilibrium, then they must satisfy:

$$p_N \geq \cdots \geq p_1, q_N \geq \cdots \geq q_1$$
$$p_i - C(q_i, \theta_i) \geq p_j - C(q_j, \theta_i), \forall i, j$$

- We will ignore other deviations: with appropriate out-of-equilibrium beliefs

Feasible Signaled Qualities

- Signaled qualities: main determinant of incentives
- Discrete signal space:

$$p_i = \sum_s \pi\left(\{s\}|q_i\right) \frac{\sum_j \pi\left(\{s\}|q_j\right) f_j q_j}{\sum_j \pi\left(\{s\}|q_j\right) f_j}$$

  - Signaled qualities: weighted average of qualities
- First Key Property:
  - Equal in expectation:

$$\sum_i f_i p_i = \sum_i f_i q_i$$

    - Equivalent to standard Bayes plausibility

Feasible Signaled Qualities

- Signaled quality of the lowest type

$$p_1 = \sum_j a_{1j} q_j \geq q_1$$

with equality if and only if θ_1 sends signals separate from everyone else

- What about the two lowest types?

$$\frac{f_1 p_1 + f_2 p_2}{f_1 + f_2} \underset{?}{\lesseqgtr} \frac{f_1 q_1 + f_2 q_2}{f_1 + f_2}$$

  - If the signals from $\{\theta_1, \theta_2\}$ separate from everyone else: holds with equality
  - If the signals overlap: put some weight on higher qualities

$$\frac{f_1 p_1 + f_2 p_2}{f_1 + f_2} > \frac{f_1 q_1 + f_2 q_2}{f_1 + f_2}$$

Feasible Signaled Qualities

- Extend this insight: theory of majorization a la Hardy, Littlewood and Polya (1934)

Definition. $\mathbf{p}$ F- majorizes $\mathbf{q}$ or $\mathbf{p} \succcurlyeq_F \mathbf{q}$ if

$$\sum_{i=1}^{k} f_i p_i \geq \sum_{i=1}^{k} f_i q_i, \forall k = 1, \cdots N-1$$

$$\sum_{i=1}^{N} f_i p_i = \sum_{i=1}^{N} f_i q_i$$

Majorization: Basic Properties

- $\mathbf{p} \succcurlyeq_F \mathbf{q}$: roughly saying that dispersion in $\mathbf{p}$ < dispersion in $\mathbf{q}$
- Relationship with Second Order Stochastic Dominance:
  - When $N = 2$: equivalent to SOSD

$$p_2 - p_1 \leq q_2 - q_1$$

  - When $N = 3$: it is also equivalent to SOSD
  - When $N \geq 4$: it is not!
  - If $\mathbf{p} \succcurlyeq_F \mathbf{q}$, then $(\mathbf{p}, F) \succcurlyeq_{SOSD} (\mathbf{q}, F)$. Reverse is not true. Stronger notion than SOSD.

Majorization: Basic Properties

- $\succcurlyeq_F$ is transitive.
- The set of $\mathbf{p}$ that F-majorize $\mathbf{q}$ is convex.
- Can show that there exists a positive matrix $\mathbf{A}$ such that $\mathbf{p} = \mathbf{A}\mathbf{q}$ where

$$\mathbf{f}^T \mathbf{A} = \mathbf{f}^T, \mathbf{A}\mathbf{e} = \mathbf{e}$$

  with $\mathbf{e} = (1, \cdots, 1)$ and $\mathbf{f} = (f_1, \cdots, f_N)$.
- We refer to $\mathbf{A}$ as an F-stochastic matrix.
  - Set of F-stochastic matrices is closed under matrix multiplication.

Majorization: Main Result

Theorem. Consider vectors of signaled and true qualities, $\mathbf{p}, \mathbf{q}$ and suppose that they satisfy

$$p_1 \leq \cdots \leq p_N, q_1 \leq \cdots \leq q_N$$

where equality in one implies the other. Then $\mathbf{p} \succcurlyeq_F \mathbf{q}$ if and only if there exists a rating system (π, S) so that

$$p_i = \mathbb{E}\left[\mathbb{E}\left[q|s\right]|q_i\right]$$

Majorization: Proof of The Main Result

- First direction: If $p_i = \mathbb{E}\left[\mathbb{E}\left[q|s\right]|q_i\right]$, then an argument similar to the above can be used to show that $\mathbf{p} \succcurlyeq_F \mathbf{q}$.
  - If all states below k have separate signals from those above, then $\sum_{i=1}^{k} f_i p_i = \sum_{i=1}^{k} f_i q_i$.
  - With overlap, $\sum_{i=1}^{k} f_i p_i$ can only go up.

Majorization: Proof of The Main Result

- Second direction:
  - First step: show that the set of signaled qualities $\mathcal{S}$ is convex ▸ Proof
  - Illustration for $N = 2$.

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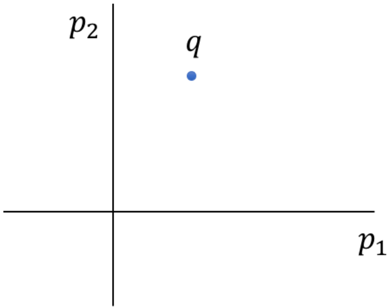

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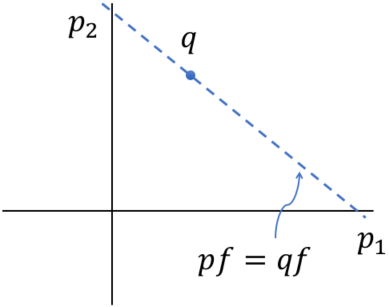

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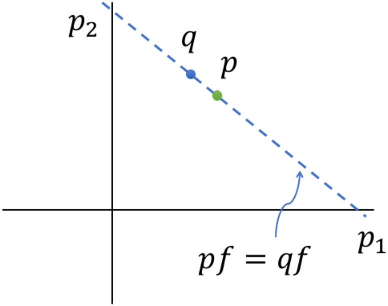

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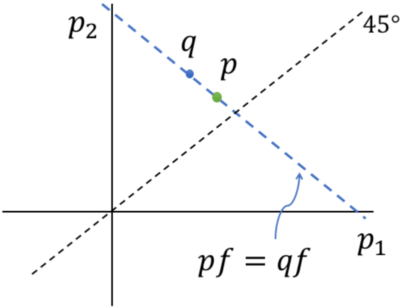

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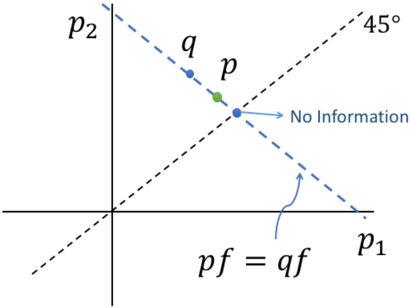

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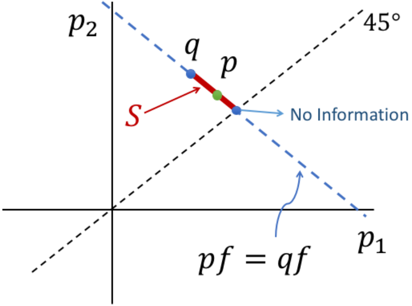

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  - Second step: For higher dimensions, for every direction $\boldsymbol{\lambda} \neq \mathbf{0}$, find two points in S, $\tilde{\mathbf{p}}_1$ and $\tilde{\mathbf{p}}_2$ such that

$$\boldsymbol{\lambda} \cdot \tilde{\mathbf{p}}_1 \leq \boldsymbol{\lambda} \cdot \mathbf{p} \leq \boldsymbol{\lambda} \cdot \tilde{\mathbf{p}}_2$$

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  - Since $\mathcal{S}$ is convex, separating hyperplane theorem implies that $\mathbf{p}$ must belong to $\mathcal{S}$.

Majorization: Continuous Case

- Proof of Theorem 1 uses induction
- We can extend the results to the case with continuous distribution
  - Use the fact that discrete distributions are dense in the space of distributions (and that the space of distributions over distributions is compact!!)

Majorization: Continuous Case

- Proof of Theorem 1 uses induction
- We can extend the results to the case with continuous distribution
  - Use the fact that discrete distributions are dense in the space of distributions (and that the space of distributions over distributions is compact!!)
- We say $p(\cdot) \succcurlyeq_F q(\cdot)$ if

$$\int_{\underline{\theta}}^{\theta} p(\theta') \, dF(\theta') \geq \int_{\underline{\theta}}^{\theta} q(\theta') \, dF(\theta'), \forall \theta \in Supp(F)$$

$$\int_{\underline{\theta}}^{\overline{\theta}} p(\theta) \, dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} q(\theta) \, dF(\theta)$$

Majorization: Continuous Case

Corollary. Let $p(\theta)$ and $q(\theta)$ be two functions representing signaled and true quality. Then, these functions arise from an equilibrium for some rating system if and only if they satisfy the following:

1. The surplus function $S(\theta) = p(\theta) - C(q(\theta), \theta)$ is differentiable and satisfies

$$S'(\theta) = -C_\theta(q(\theta), \theta)$$

2. The functions $p(\theta)$ and $q(\theta)$ are increasing in θ and satisfy $p \succcurlyeq_F q$.

Constructing Signals

- In general, characterizing an RS that leads to a signaled quality is very difficult.
- Some partial characterizations:

Definition. A Rating System is *separating at* θ, if the signals generated by $\{q(\theta') : \theta' \leq \theta\}$ does not overlap with those generated by $\{q(\theta') : \theta' > \theta\}$, almost-surely.

Constructing Signals

- Separation occurs when majorization constraint binds

Corollary. Let $p(\theta)$ and $q(\theta)$ be a pair of signaled and true quality functions that satisfy $p \succcurlyeq_F q$. Let (π, S) be a rating system for which $p(\theta) = \mathbb{E}[\mathbb{E}[q|s]|q(\theta)]$. Then (π, S) is separating at θ if and only if the majorization inequality binds at θ.

Constructing Signals

- One easy case: $p(\theta)$ flatter than $q(\theta)$, i.e., $p'(\theta) < q'(\theta)$
  - majorization constraint never binds.

- Signal:

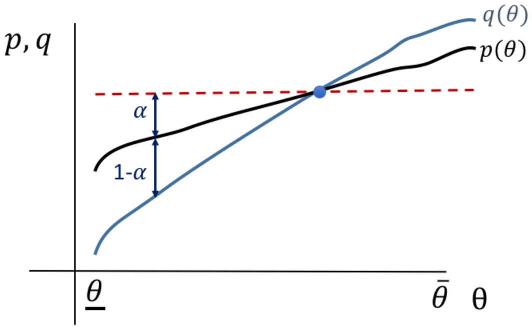
$$S = \{q(\theta) : \theta \in \Theta\} \cup \{\emptyset\}$$

$$\pi(\{s\} | q) = \begin{cases} \alpha(q) & s = q \\ 1 - \alpha(q) & s = \emptyset \end{cases}$$

  - Reveal quality or say nothing!

Non-separating signal

When $p(\theta)$ is flatter than $q(\theta)$



Constructing Signals: Algorithm

- For the discrete case, we can give an algorithm to construct the signals (rough idea; much more details in the actual proof)
  1. Start from $\mathbf{q}$
  2. Consider a convex combination of two signals:
     2.1 Full revelation: $\pi^{FI}(\{q\}|q) = 1$
     2.2 Pooling signal: pool two qualities q_i and q_j

$$S = \{q_1, \cdots, q_N\} - \{q_i, q_j\} \cup \{q_{ij}\}$$

$$\pi^{i,j}(\{s\}|q) = \begin{cases} 1 & s = q, q \neq q_i, q_j \\ 1 & s = q_{ij}, q = q_i, q_j \end{cases}$$

     2.3 Send π^{FI} with probability α and $\pi^{i,j}$ with probability $1 - \alpha$
  3. Choose α so that the resulting signaled quality has one element in common with $\mathbf{p}$
  4. Repeat the same procedure on resulting signaled quality until reaching $\mathbf{p}$

Optimal Rating Systems

- So far: characterized the set of all feasible allocations
- Now: Pareto optimal allocations
- Approach: fix u, maximize $\int \lambda(\theta)\,\Pi(\theta)\,dF(\theta)$
  - $\lambda(\theta) = 1$: Total surplus
  - $\lambda(\theta) = \delta(\theta - \underline{\theta})$: Buyer optimal
  - $\lambda(\theta)$: increasing; higher weight on higher quality sellers

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- Alternative - harder: characterize pareto frontier with payoffs

Total Surplus

- First Best allocation: maximizes total surplus ignoring all the constraints

$$C_q\left(q^{FB}(\theta),\theta\right)=1$$

- Incentive constraint:

$$p'(\theta)=C_q(q(\theta),\theta)\,q'(\theta)=q'(\theta)$$

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- Set $p(\theta)=q(\theta)$
    - Satisfies IC
    - Satisfies majorization

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  - Satisfies majorization
- Maximizing total surplus: full information about quality

Buyer Optimal Allocations

- Suppose that $\lambda(\theta) = \delta(\theta - \underline{\theta})$: Dirac's delta
  - Textbook mechanism design problem: all types have the same outside option; only binding for the lowest type

Buyer Optimal Allocations

- Suppose that $\lambda(\theta) = \delta(\theta - \underline{\theta})$: Dirac's delta
  - Textbook mechanism design problem: all types have the same outside option; only binding for the lowest type
- Tradeoff: information rents vs. reallocation of profits
  - Want to allocate resources to the lowest type and the buyers
  - All higher quality types want to lie downward
- Reduce qualities relative to First Best

Buyer Optimal Allocations

Relaxed Mechanism Design problem - ignore majorization constraint

$$\max \Pi\left(\underline{\theta}\right)$$

subject to

$$\Pi'\left(\theta\right) = -C_{\theta}\left(q\left(\theta\right), \theta\right)$$

$$q\left(\theta\right) : \text{increasing}$$

$$u + \int_{\underline{\theta}}^{\overline{\theta}} \Pi\left(\theta\right) dF\left(\theta\right) = \int_{\underline{\theta}}^{\overline{\theta}} \left[q\left(\theta\right) - C\left(q\left(\theta\right), \theta\right)\right] dF\left(\theta\right)$$

Proposition. A quality allocation $q\left(\theta\right)$ is buyer optimal if and only if it is a solution to the relaxed problem. Moreover, if the cost function $C\left(\cdot, \cdot\right)$ is strictly submodular, then a buyer optimal rating system never features a separation.

Buyer Optimal Allocations: Intuition

- The solution of the relaxed problem

$$C_q\left(q\left(\theta\right),\theta\right) \leq 1$$

- Incentive constraint

$$p'\left(\theta\right) = C_q\left(q\left(\theta\right),\theta\right) q'\left(\theta\right)$$

- $p\left(\theta\right)$ flatter than $q\left(\theta\right)$: majorization constraint never binds
  - If $C_q < 1$ for a positive measure of types, no separation of qualities

Buyer Optimal Allocations: Extension

Corollary. If $\lambda(\theta)$ is decreasing in θ, then the majorization inequality is slack at the optimum. Furthermore, if $C(\cdot,\cdot)$ is strictly sub-modular, then the optimal rating system never features any separation.

High Quality Seller Optimal

- Now suppose $\lambda(\theta)$ is increasing in θ
- Solution of the relaxed mechanism design problem satisfies

$$C_q(q(\theta), \theta) \geq 1$$

- IC:

$$p'(\theta) = C_q(q(\theta), \theta) q'(\theta) > q'(\theta)$$

High Quality Seller Optimal

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$$C_q(q(\theta), \theta) \geq 1$$

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$$p'(\theta) = C_q(q(\theta), \theta) q'(\theta) > q'(\theta)$$

- Majorization inequality will be violated
  - Intuition: overprovision of quality to prevent low θ's from lying upwards; signaled quality must be steep

High Quality Seller Optimal

Proposition. Suppose that $\lambda(\theta)$ is increasing and $\lambda\left(\overline{\theta}\right) > \lambda\left(\underline{\theta}\right)$. Then optimal rating system must feature separation. Furthermore, optimal qualities must feature a jump.

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- Intuition:
  - No jumps $\Rightarrow$ majorization binds over only over intervals:

$$p(\theta) = q(\theta)$$

  - Incentive constraint:

$$p'(\theta) = C_q(q(\theta), \theta) q'(\theta) \rightarrow C_q(q(\theta), \theta) = 1$$

  - Outside of the interval

$$C_q > 1$$

  - Majorization is violated

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$$C_q > 1$$

  - Majorization is violated
- Similar feature to partition information

Conclusion

- Rating Systems in a competitive model of adverse selection and moral hazard
- Provide full characterization of feasible allocations:
  - Majorization
- Pareto optimal rating systems

Convexity of $\mathcal{S}$

- Discrete signal space:

$$p_i = \sum_s \pi\left(\{s\}|q_i\right) \frac{\sum_j \pi\left(\{s\}|q_j\right) f_j q_j}{\sum_j \pi\left(\{s\}|q_j\right) f_j}$$

- Alternative representation of the RS:

$$\tau \in \Delta\left(\Delta\left(\Theta\right)\right) : \mu_j^s = \frac{\pi\left(\{s\}|q_j\right) f_j}{\sum_j \pi\left(\{s\}|q_j\right) f_j}, \tau\left(\{\boldsymbol{\mu}^s\}\right) = \sum_j \pi\left(\{s\}|q_j\right) f_j$$

- Bayes plausibility

$$\mathbf{f} = \int_{\Delta(\Theta)} \boldsymbol{\mu} d\tau$$

- We can write signaled quality as

$$\mathbf{p} = \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \mathbf{q} = \mathbf{A}\mathbf{q}$$

Convexity of $\mathcal{S}$

- The set $\mathcal{S}$ is given by

$$\mathcal{S} = \left\{ \mathbf{p} : \exists \tau \in \Delta\left(\Delta\left(\Theta\right)\right), \int \boldsymbol{\mu} d\tau = \mathbf{f}, \mathbf{p} = \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \right\}$$

- For any τ_1, τ_2 satisfying Bayes plausibility, i.e., $\int \boldsymbol{\mu} d\tau = \mathbf{f}$,their convex combination also satisfies BP since integration is a linear operator.
- Therefore

$$\begin{aligned} \lambda \mathbf{p}_1 + (1-\lambda)\mathbf{p}_2 = \lambda \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau_1 + \\ (1-\lambda)\,\text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau_2 \\ = \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\left(\lambda\tau_1 + (1-\lambda)\tau_2\right) \end{aligned}$$

- Since $\lambda\tau_1 + (1-\lambda)\tau_2$ satisfies BP, $\lambda \mathbf{p}_1 + (1-\lambda)\mathbf{p}_2 \in \mathcal{S}$

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