Optimal Rating Design[*]

Maryam Saeedi
Carnegie Mellon University
msaeedi@andrew.cmu.edu

Ali Shourideh
Carnegie Mellon University
ashourid@andrew.cmu.edu

August 21, 2020

Abstract

We study the design of optimal rating systems in the presence of adverse selection and moral hazard. Buyers and sellers interact in a competitive market where goods are vertically differentiated according to their qualities. Sellers differ in their cost of quality provision, which is private information to them. An intermediary observes sellers' quality and chooses a rating system, i.e., a signal of quality for buyers, in order to incentivize sellers to produce high-quality goods. We provide a full characterization of the set of payoffs and qualities that can arise in equilibrium under an arbitrary rating system. We use this characterization to analyze Pareto optimal rating systems when seller's quality choice is deterministic and random.

[*]We thank James Best, Aislinn Bohren, Odilon Camara, Emir Kamenica, Alexey Kushnir, Jacobo Perego, and Ilya Segal as well as various seminar and conference participants for their helpful comments.

1 Introduction

The problem of information control is at the heart of the design of markets with asymmetric information. On platforms such as Airbnb or eBay, where buyers often find it difficult to evaluate the quality of the service or product being offered, information provided by the platform can mitigate some of the problems arising from adverse selection and moral hazard. In insurance markets, where providers do not have precise information about insurees and want to condition contracts on public signals, regulators control which information can be used by providers. This centrally provided information can be used to incentivize the parties involved and improve allocative efficiency in the market. In this paper, we study the design of information control – henceforth rating systems – in markets with adverse selection and moral hazard.

We perform this exercise in a competitive model with adverse selection and moral hazard. In our model, seller types are privately known, and each seller produces a single product that is vertically differentiated by its quality. Producing higher-quality goods is costly for the sellers, but it is less so for higher-type sellers than for lower-type ones. Buyers are uninformed about the quality of the product sold in the market and have to rely on the information provided by an intermediary, who observes product quality and sends a signal to buyers. We refer to such signal structure as a rating system.

Our goal in this paper is twofold. First, we characterize the allocations of qualities that are achievable by an arbitrary rating system (i.e., implementable allocations). Second, we describe the properties of optimal rating systems when quality outcomes are deterministic and random. In our analysis, optimality of rating systems are evaluated according objectives associated with Pareto optimality, i.e., those that maximize a weighted average of buyers' and sellers' payoffs, as well as the revenue earned by the intermediary.

Buyers use the information provided by the intermediary to form expectations about the quality of the goods in the market. This information also impacts sellers' choice of quality. In particular, on the one hand, sellers' incentives for choosing a quality level are affected by how this choice affects the expected prices. On the other hand, since all buyers value quality equally, their posterior belief about sellers' quality is reflected in product prices. Therefore, the main determinant of sellers' incentives is their (second-order) belief about buyers' posterior beliefs after observing their signal. We refer to these second-order expectations as sellers' signaled qualities. As a result, characterizing the set of payoffs and allocations boils down to characterizing these second-order expectations. This is in addition to the standard notion of incentive compatibility, which is associated with the optimal choice of quality by sellers.

Our main implementability result is that these second-order expectations are related to the chosen qualities via a weighted-majorization ranking. In particular, we show that if type θ sellers (distributed according to F and ranked by their efficiency of quality provision) choose quality $q(\theta)$, then their expectation of buyers' expectation (upon realizing a signal), $q(\theta)$, must F-majorize $\overline{q}(\theta)$; in other words

$$\int_{\underline{\theta}}^{\theta} \overline{q}(\theta)\, dF(\theta) \geq \int_{\underline{\theta}}^{\theta} q(\theta)\, dF(\theta), \forall \theta$$

with equality for the highest type. The reverse is also true; that is, if $q(\theta)$ F-majorizes $\overline{q}(\theta)$, then a rating system can be constructed so that $\overline{q}(\theta)$ is the expected value of buyers' average posterior from the perspective of a type θ seller.

This ranking of the two functions, $\overline{q}(\theta)$ and $q(\theta)$, is equivalent to the standard notion of mean preserving spread. In particular, it can be shown that $q(\cdot)$ $F-$majorizes $\overline{q}(\cdot)$ if and only if the random variable induced by $q(\cdot)$ using $F(\cdot)$ is a mean preserving spread of that induced by $q(\cdot)$. This result, thus, can be regarded as an extension of Blackwell (1953)'s result on the relationship between the distribution of the posterior mean and the prior for an arbitrary signal structure. Recall that in our setup, since we need to consider sellers' incentives in choosing quality, the characterization of the posterior mean is not sufficient. We have, thus, shown that the expectation of the posterior mean conditional on the state (second-order expectations or signaled qualities) also satisfies the mean preserving spread property of Blackwell (1953).

Using the majorization formulation has two benefits: first, it allows us to work with functions of type θ sellers as opposed to distributions; second, when this inequality is binding for a certain type θ, the equivalent rating system must separate sellers with quality below $q(\theta)$ from those with higher quality, so their signals must not overlap. In other words, the signals sent by the types below θ and those sent by the types with higher quality must not overlap. Additionally, we provide an algorithm that constructs a rating system which implements any schedule of second-order expectations (signaled qualities) and true qualities when the type space is discrete. For continuous type spaces, this algorithm can be applied by approximating continuous distributions with a discrete one.

This characterization of the set of implementable qualities and signaled qualities (i.e., sellers' second-order expectations) allows us to cast the problem of optimal rating design as a mechanism design problem. In particular, an allocation of qualities and signaled qualities is implementable if and only if it satisfies the standard notion of incentive compatibility together with the majorization ranking described above. If we interpret signaled qualities as transfers, the problem of optimal rating design is equivalent to the standard mechanism design with transferrable utility – such as that of Mussa and Rosen (1978), Baron and Myerson (1982), and Myerson (1981), among many others – where transfers have to satisfy a dispersion constraint implied by the majorization ranking. In the second part of the paper, we provide techniques to solve a subset of such mechanism design problems, which are not convex optimization problems, because of the presence of the majorization constraint. More specifically, we study two versions of our model: one in which quality outcomes for buyers are deterministic and one in which they are random.

When chosen qualities are deterministic, the main classes of objectives that we consider are the weighted sum of sellers' payoffs. In particular, we allow for sellers' welfare weights to depend on their cost type. We consider three classes of rating systems: (1) low-quality optimal, in which the welfare weights are decreasing in sellers' types ; (2) high-quality-seller optimal, in which welfare weights increase with θ (i.e., the sellers with a lower cost of quality provision have a higher weight); and (3) mid-quality-seller optimal, in which welfare weights are hump-shaped in θ.[1]

There are a two main insights that arise from the analysis of the deterministic case. First, random messages are a robust feature of optimal rating systems. This is mainly due to the fact that quality outcomes can be fully controlled by sellers. As a result, if rating systems are restricted to be deterministic this necessarily leads to bunching of types and jumps in quality choices. To the extent that for a wide range of welfare weights, jumps in quality choices are not desired, mixing must be introduced. In the class of objectives considered, the optimal rating systems takes the form of what we call full mixing; one in which the quality is revealed with some probability and otherwise a generic message – common to all quality levels – is sent.

Second, information revelation interacts with whether welfare weights are increasing or decreasing in seller quality, i.e., θ. Increasing welfare weights creates a motive for profits to be reallocated to higher quality types. We show that in this case, the best that can be done is to reveal all information. Note that when welfare weights are increasing, a mechanism designer that has access to unrestricted monetary transfers reallocates profits by marginally compensating choice of quality by more than one-to-one. When the rating system is incentivizing sellers, the majorization result constrains the rewards for quality. More specifically, since dispersion of signaled qualities are less than that of chosen qualities, they cannot create steep rewards for choice of quality by sellers. As a result, the best that can be done is fully reveal quality. We establish this intuition via series of perturbation arguments in our mechanism design problem.

In contrast, when the welfare weights are decreasing in seller quality, use of random mixing is optimal. In this case, partial pooling of quality choices reduces the reward for choice of quality and allows profits to be reallocated to lower types. Consideration of mid-quality-seller optimal ratings confirms this insight. When welfare weights are hump-shaped in θ, optimal rating system reveals all information for low qualities while it is full mixing for middle and high qualities.

While probabilistic information structures are a common feature of Pareto optimal ratings, they are somewhat uncommon in reality. We show that when quality outcomes are random probabilistic information structures are not needed. That is, the mixing that is required in the deterministic model can be generated with monotone partitions. We consider a version of our model where buyers' quality outcomes cannot be directly controlled by the seller; the sellers control the mean of the outcome while their cost depends on their type and the mean. We focus on montone rating systems – those in which higher quality outcomes lead to higher signaled qualities. In this environment, we show that optimal ratings always take the form of monotone partitions where qualities are either fully revealed or pooled with an interval around them. While in general characterization of optimal ratings is more difficult, we develop a mathematical result that helps us provide partial characterization of optimal ratings. We, then use this result to show that in a two type case, information must be fully revealed for extreme realizations while it should pooled for intermediate realizations.

[1]This is similar to Dworczak et al. (forthcoming) where they consider a mechanism design problem with arbitrary Pareto weights.

Finally, we consider the case of a revenue maximizing intermediary that charges the sellers a flat fee for entering the market. The existence of such a fee leads to an inefficiency since some sellers do not enter the market; only sellers with low enough cost, i.e., high enough θ enter. The rating system affects this entry margin by affecting the sellers' profits. Note that conditional on cutoff type for entry, the intermediary's revenue is maximized when the payoff of the cutoff type is maximized. This would allow the intermediary to charge a higher fee and increase its revenue. Thus, optimal rating system takes a full mixing form.

1.1 Related Literature

Our paper is related to a few strands of literature in information economics and mechanism design. Most closely, it is related to the Bayesian persuasion literature, as in Kamenica and Gentzkow (2011), Rayo and Segal (2010), Alonso and Câmara (2016), and Dworczak and Martini (2019), among many others. However, unlike most of this literature, in our setup, the state in which an information structure is designed upon is itself endogenous. As a result, the informed party's decision is affected by the choice of information structure, whereas in much of the Bayesian persuasion literature an uninformed receiver is taking an action. A notable exception is the paper by Boleslavsky and Kim (2020) where they consider a model with moral hazard where an agent controls the distribution of state with her effort. They show that Kamenica and Gentzkow (2011)'s concavification method extends to their environment. In our setup, we are able to provide a sharp characterization of the set of implementable outcomes. Furthermore, we are able to solve the resulting mechanism design problem under fairly general assumptions on the cost function and distribution of types. Kolotilin et al. (2017) study a problem of information transmission where one of the parties is privately informed. However, in their setup, the informed party possesses information about her payoff which is independent of the state. In contrast, in our model sellers are informed about the state (their cost type), and the information disclosure affects their choice of quality.[2]

From a technical perspective, our paper is also related to a subset of the Bayesian persuasion literature that studies problems in which receivers' actions depend on their posterior mean. For example, Gentzkow and Kamenica (2016), Kolotilin (2018), Dworczak and Martini (2019), and Roesler and Szentes (2017) use Blackwell (1953)'s result that the existence of an information structure is equivalent to the distribution of the posterior mean second-order stochastically dominating (SOSD) the prior. However, in our study finding this posterior mean is not enough, since sellers' incentives depend on the expected prices, which are themselves determined by the expectation of the posterior mean conditional on the state. Our contribution to this literature is to show that any profile of second-order expectations that dominates the chosen qualities in the sense of Second Order Stochastic Dominance can be derived from some information structure. Moreover, we use the majorization ranking

[2] Few other papers have also focused on the joint problem of mechanism and information design; Guo and Shmaya (2019) and Doval and Skreta (2019) are notable examples.

in order to shed light on key properties of all the information structures that induce a certain distribution of second-order expectations.[3] In our formulation, we use the majorization ranking for the functions representing quality and signaled quality – second order expectations. Thus our mechanism design problem is equivalent to a mechanism design problem with transfers in which the transfer function majorizes the quality function. Similar to this problem, Kleiner et al. (2020) solve a class of problems where majorization appears as a constraint. Their solution method uses the characterization of extreme points of the set of functions that majorizes a certain function. In contrast, our solution of the mechanism design problem involves calculus of variations due to the lack of linearity that is present in their model.[4]

Our paper is also related to the extensive literature on contracting and mechanism design. Where as often the main assumption is that monetary transfers are available to provide incentives, in our setup incentives for quality provision are provided using the rating system. In fact, this is often the case in multi-sided platforms: rider and driver ratings in Uber and Lyft and seller badges in eBay and Airbnb are a few examples.[5] A few notable exceptions are models that study the problem of certification and its interactions with moral hazard: Albano and Lizzeri (2001), Zubrickas (2015), and Zapechelnyuk (2020).[6] An important contribution is that of Albano and Lizzeri (2001) where a key assumption is that the intermediary can charge an arbitrary fee schedule. The presence of an unrestricted fee schedule potentially reduces the importance of the certification mechanism. This is in contrast with our model where monetary transfers are not flexible. More recently, Zubrickas (2015) and Zapechelnyuk (2020) also study variants of this problem. Their focus is, however, on deterministic ratings. As we show, random signals are an important feature of optimal mechanisms. Additionally, we analyze ratings when qualities are random and not fully controlled by the providers.

The rest of the paper is organized as follows: in section 2 we set up the model; in section 3, we describe the set of implementable allocations; in section 4 we describe Pareto-optimal rating systems with deterministic quality choice, in section 5 we analyze the model with random qualities, and finally in section 6 we consider some extensions of our model including the problem of a revenue-maximizing intermediary.

[3] Dworczak and Martini (2019) develop a methodology akin to duality to solve a large class of such problems. However, their methods do not apply to our case, as our resulting mechanism design problem is non-convex. Thus, we have to use perturbation techniques, as described in section 4, to verify solutions.

[4] Gershkov et al. (2020) study optimal auction design with risk-averse bidders who have dual risk aversion a la Yaari (1987). In their problem, the feasibility of allocations implies a majorization constraint on quantities, i.e., probability of allocation of the object to each bidder. Similar to our paper, they use calculus of variations to solve this problem. In contrast, our mechanism design problem is equivalent to a problem in which transfers must be majorized by qualities. This together with incentive compatibility puts more restriction on the set of implementable allocations.

[5] Higher value of these ratings could lead to priority in receiving the service while low values could lead to exclusion from the platform.

[6] Evidently, our paper is also related to the extensive and growing literature that studies the problem of certification and information disclosure (e.g., Lizzeri (1999), Ostrovsky and Schwarz (2010), Boleslavsky and Cotton (2015), Harbaugh and Rasmusen (2018), and Hopenhayn and Saeedi (2020)).

2 The Model

In this section, we describe our baseline model of adverse selection and moral hazard that will provide the main framework for our analysis. We consider an economy with a continuum of sellers and buyers.[7] Each seller produces a single product that is vertically differentiated by quality. Upon making a purchase, the buyer evaluates the good according to the following payoff function:

$$q - t,$$

where q is the quality of the good produced and t is the transfer made to the seller.

Sellers choose whether to produce or not and at which quality level. The cost of producing a good with quality q is given by $C(q, \theta)$, where θ is the type of seller. We assume that θ is drawn from a distribution with a c.d.f. given by $F(\theta)$ and support Θ. We allow $F(\cdot)$ to be a piecewise continuous function with a finite set of discontinuity points; that is, $F(\cdot)$ could be a mixture of a continuous and (finite) discrete distribution.

We make the following assumptions on the cost function:

Assumption 1. *The function $C(q; \theta)$ satisfies $C_q \geq 0$, $C_\theta \leq 0$, $C_{qq} \geq 0$, and $C_{q\theta} \leq 0$. Moreover, $C(0, \theta) = 0$ and $C_q(0, \theta) = 0$.*

The submodularity assumption on $C(\cdot, \cdot)$, $C_{\theta q} \leq 0$, ensures that it is efficient to have higher θ's to produce higher-quality goods; that is, higher values of θ have a lower marginal cost of producing higher-quality goods. Finally, sellers' payoffs are given by

$$t - C(q, \theta),$$

where t is the transfer they receive.

For simplicity, we normalize the outside option of buyers and sellers to 0. In the first part of our analysis, we focus on the cases where all sellers produce. In section 6, we discuss various assumptions about the entry of both buyers and sellers into the market.

We assume that buyers are uninformed about the quality of the product sold in the market and have to rely on the information provided by an intermediary, who observes the quality of the products sold by each seller and sends a partially informative signal to buyers. This is represented by an information structure or experiment a la Blackwell (1953) and is given by a signal space S and probability measure $\pi(\cdot|q) \in \Delta(S)$. We refer to this information structure (π, S) as a rating system. One can interpret this assumption on the information structure in multiple ways. One interpretation is that of a platform which observes certain information about sellers' past behavior and uses aggregated signals to provide information to buyers. Another interpretation is that of a regulator who regulates the information that can be used in contracts. Such regulations are fairly common in insurance markets. For example, community ratings in health insurance markets restrict the extent to which insurance rates can vary across individuals.

[7] Alternatively, we can think of this as an economy with one seller and one buyer. While this setup is mathematically equivalent to ours, the assumption of perfect competition is easier to interpret with a large number of sellers and buyers.

Given the information provided by the intermediary, buyers form expectations about the quality of the goods in the market and compete over them. Since the only information buyers observe about the products is the signal $s \in S$ provided by the intermediary, there is a price $p(s)$ for each signal realization. In our baseline model, we assume that buyers compete away their surplus and the price for each signal realization satisfies

$$p(s) = \mathbb{E}[q|s],$$

where the conditional expectation is taken using a prior on the distribution of qualities chosen by the seller and the signal structure by the intermediary. Thus, our assumption is that buyers know the signal structure together with the strategies used by sellers in terms of their quality choices.[8]

A seller of type θ that chooses a quality level q' has the following payoff:

$$\int_S p(s)\,\pi(ds|q') - C(q',\theta),$$

where $\int_S p(s)\,\pi(ds|q')$ is the expected price received by the seller. In other words, sellers must take into account the fact that upon choosing a quality, there will be a distribution over the posteriors formed by buyers, which in turn affect the prices they face. Simply put, sellers' payoffs depend on their beliefs about buyers' beliefs (i.e., their second-order beliefs).

Hence, given a rating system (π, S), equilibrium quality choices $\{q(\theta)\}_{\theta\in\Theta}$ by the sellers must satisfy the following incentive compatibility constraint

$$q(\theta) \in \arg\max_{q'} \int_S \mathbb{E}[q|s]\,\pi(ds|q') - C(q',\theta); \tag{1}$$

together with their participation constraint:

$$\max_{q'} \int_S \mathbb{E}[q|s]\,\pi(ds|q') - C(q',\theta) \geq 0.$$

We define a seller's *signaled quality* as

$$\overline{q}(\theta) = \int_S \mathbb{E}[q|s]\,\pi(ds|q(\theta)). \tag{2}$$

As an example, when signals are deterministic and $\pi(\cdot|q)$ is degenerate, then signaled quality, $\overline{q}(\theta)$, is the average quality among the sellers who send the same signal as θ. Signaled qualities and their dependence on θ are the main determinants of the sellers' incentives to choose their desired level of quality.

The above definition of equilibrium for an arbitrary rating system or information structure clarifies the key difference between our setting and that of the models of persuasion a

[8] In section 6 we consider alternative determination of prices. The zero surplus assumption for the buyers is made out of convenience and not necessary for the analysis. What is necessary is that buyers' surplus is equated across signal realizations.

la Kamenica and Gentzkow (2011). What differentiates our setup from Bayesian persuasion is the fact that due to moral hazard (i.e., q is a sellers' choice) the state is endogenous to the information structure. As we show in section 3, this endogeneity leads to incentive compatibility (as it does in mechanism design) and characterization of second-order expectations. We then use this characterization to describe the properties of optimal rating systems.

3 Characterization of General Rating Systems

In this section, we provide a characterization of the set of payoffs and qualities that can be achieved in equilibrium by any rating system. The analysis sheds light on the restrictions that are imposed by the particular way that incentives are provided via the rating system.

3.1 Discrete Distribution of Sellers' Types

We start our analysis by first assuming that the distribution of sellers' types is discrete: $\Theta = \{\theta_1 < \cdots < \theta_N\}$ and f_i is the probability that a seller's type is θ_i (we still refer to the distribution θ as F). As it is convenient to use a vector notation to describe allocations, we describe the distribution of θ by its vector of point mass function $\mathbf{f} = (f_1, \cdots, f_N)$. Additionally, the vector of qualities and signaled qualities is given by $\mathbf{q} = (q_1, \cdots, q_N)$ and $\overline{\mathbf{q}} = (\overline{q}_1, \cdots, \overline{q}_N)$, respectively, where q_i is the quality chosen by a seller of type θ_i and $\overline{q}_i$ is her signaled quality implied by the rating system. Throughout our analysis, vectors are column vectors and are row vectors when they are transposed (e.g., $\mathbf{q}$ is a column vector and $\mathbf{q}^T$ is a row vector).

As we discuss below, the problem of characterizing the equilibrium payoffs and qualities for arbitrary information structures boils down to the characterization of the set of possible signaled qualities $\overline{\mathbf{q}}$ for a given allocation of quality $\mathbf{q}$. In what follows, we proceed towards a full characterization of this set.

We first establish that the main determinant of sellers' incentives are signaled qualities, $\overline{\mathbf{q}}$. To see this, note that since a seller of type θ_i can choose q_j, then incentive compatibility (1) implies that

$$\overline{q}_i - C(q_i, \theta_i) \geq \overline{q}_j - C(q_j, \theta_i), \forall j, i.$$

We can resort to an argument in the spirit of the revelation principle and use the above inequalities in place of the constraint (1). This is mainly because we can always choose a particular signal $s_\emptyset$ to be associated with off-path qualities together with buyers' belief that the quality associated with such a signal is 0.[9] This would imply that by deviating to a quality other than q_j, prices will be 0. Therefore, the above constraint is equivalent to the constraint in (1).

The following lemma characterizes standard incentive compatibility:

Lemma 1. *If a vector of qualities,* $\mathbf{q}$*, and a vector of signaled qualities,* $\overline{\mathbf{q}}$*, arise from an equilibrium, then they must satisfy:*

[9]Note that the off-path qualities and buyers' belief are not pinned down for off-path values of qualities.

$$\overline{q}_N \geq \cdots \geq \overline{q}_1, q_N \geq \cdots \geq q_1$$
$$\overline{q}_i - C(q_i, \theta_i) \geq \overline{q}_j - C(q_j, \theta_i), \forall i, j.$$

The proof is standard and is omitted.

Lemma (1) establishes that signaled qualities, $\overline{q}_i$'s, paired with chosen qualities, q_i, must be incentive compatible. A natural question then arises: Does the fact that $\overline{q}_i$'s are derived from second-order expectations in (2) impose any restrictions on them? In what follows, we show that the fact that signaled qualities are derived from second-order expectations for an arbitrary rating system is equivalent to second-order stochastic dominance. That is, it is equivalent to the random variable implied by $\overline{q}_i$ distributed according to F dominating the random variable induced by q_i distributed according to F in the sense of second-order stochastic dominance.

In formulating second-order stochastic dominance, we use an alternative to the familiar formulation of Rothschild and Stiglitz (1970). In particular, we use the majorization formulation of second-order stochastic dominance, which, as we show later, allows us to provide a sharp characterization of rating systems that induces a certain distribution of signaled qualities. Our approach is based on the majorization ranking introduced by Hardy et al. (1934). See Marshall et al. (1979) for a thorough treatment of the concept.

More specifically, consider two random variables, $\mathbf{x}$ and $\mathbf{y}$, that take on real values in $\{x_1 \leq \cdots \leq x_N\}$ and $\{y_1 \leq \cdots \leq y_N\}$, respectively, and whose distribution is given by $\Pr(\mathbf{x} = x_i) = \Pr(\mathbf{y} = y_i) = f_i$. We say that $\mathbf{y} \succcurlyeq_F \mathbf{x}$ or $\mathbf{x}$ is F-majorized by $\mathbf{y}$ if the following holds:

$$\sum_{i=1}^{N} f_i x_i = \sum_{i=1}^{N} f_i y_i,$$
$$\sum_{j=1}^{i} f_j x_j \geq \sum_{j=1}^{i} f_j y_j, \forall i = 1, \cdots, N-1. \tag{3}$$

Hardy et al. (1929) showed that the above is equivalent to $\sum_{i=1}^{N} f_i u(x_i) \geq \sum_{i=1}^{N} f_i u(y_i)$ for any concave function $u(\cdot)$; that is, it is equivalent to the standard second-order stochastic dominance. As it will be clear later, we prefer this formulation of second-order stochastic dominance, since it informs us of the properties of rating systems. We follow Hardy et al. (1929) and refer to the inequalities in (3) as majorization inequalities.

In order to describe the properties of $\overline{\mathbf{q}}$ and $\mathbf{q}$, we follow Kamenica and Gentzkow (2011) and represent the rating system (π, S) as a distribution over the distribution of posteriors, $\tau \in \Delta(\Delta(\Theta))$ that satisfies the Bayes plausibility constraint, which can be written in vector form as

$$\mathbf{f} = \int_{\Delta(\Theta)} \boldsymbol{\mu} d\tau,$$

where $\boldsymbol{\mu}$ is the posterior over types – represented as a vector in $\mathbb{R}^N$. If a rating system generates a finite number of signals, then τ must have a finite support and we can construct a signal structure from τ using Bayes' rule:[10]

$$\forall \boldsymbol{\mu} \in \text{Supp}(\tau), \frac{\pi(\{s\}|q_i) f_i}{\mu_i^s} = \tau(\{\boldsymbol{\mu}^s\}),$$

where s is the signal associated with the posterior $\boldsymbol{\mu}^s$. We can thus use the above to formulate the signaled qualities as a function of actual qualities:

$$\overline{q}_i = \sum_s \pi(\{s\}|q_i) \frac{\sum_j \pi(\{s\}|q_j) f_j q_j}{\sum_j \pi(\{s\}|q_j) f_j} = \frac{1}{f_i} \int_{\mu \in \text{Supp}(\tau)} \mu_i \boldsymbol{\mu}^T \mathbf{q} d\tau \tag{4}$$

where $\boldsymbol{\mu}^T$ is the transpose of $\boldsymbol{\mu}$, and $\boldsymbol{\mu}^T\mathbf{q}$ is the inner product of $\boldsymbol{\mu}^T$ and $\mathbf{q}$. In other words, $\boldsymbol{\mu}^T\mathbf{q}$ is the posterior mean of quality, and the above integral is the expectation of the posterior mean quality from the perspective of the seller. One can write (4) in vector form as $\overline{\mathbf{q}} = \mathbf{A}\mathbf{q}$ where $\mathbf{A}$ is an $N \times N$ positive matrix which satisfies

$$\mathbf{f}^T\mathbf{A} = \mathbf{f}^T, \mathbf{A}\mathbf{e} = \mathbf{e}, \tag{5}$$

where $\mathbf{e} = (1, \cdots, 1)^T$.[11] The existence of $\mathbf{A}$ which satisfies (5) implies the following result:

Proposition 1. *Let $\overline{\mathbf{q}}, \mathbf{q}$ be vectors of signaled and true qualities, respectively, that arise in an equilibrium for some information structure. Then, $\mathbf{q} \succcurlyeq_F \overline{\mathbf{q}}$.*

The proof is relegated to the appendix.

That $\mathbf{q}$ F-majorizes $\overline{\mathbf{q}}$ is a direct result of existence of matrix $\mathbf{A}$ that satisfies (5). Loosely speaking, (5) implies that $\overline{q}_i$'s are less dispersed than q_i and thus $\overline{\mathbf{q}} \succcurlyeq_F \mathbf{q}$. Our main characterization result in this section is that the reverse of Proposition 1 holds as well:

Theorem 1. *Consider the vectors of signaled and true qualities, $\overline{\mathbf{q}}, \mathbf{q}$, respectively, and suppose that they satisfy*

$$\overline{q}_1 \leq \cdots \leq \overline{q}_N, q_1 \leq \cdots \leq q_N$$

with equality in one implying the other. Moreover, suppose that $\mathbf{q} \succcurlyeq_F \overline{\mathbf{q}}$. Then there exists a rating system (π, S) so that

$$\overline{q}_i = \mathbb{E}[\mathbb{E}[q|s]|q_i].$$

The formal proof of Theorem 1 is relegated to the appendix. Here, we provide an outline.

Consider the set of all signaled quality vectors that are induced by some rating system:

$$\mathcal{S} = \left\{ \mathbf{r} | \exists \tau \in \Delta(\Delta(\Theta)), \mathbf{r} = \text{diag}(\mathbf{f})^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \mathbf{q} \text{ with } \mathbf{f} = \int \boldsymbol{\mu} d\tau \right\}. \tag{6}$$

The set $\mathcal{S}$ is convex since for any two measures τ_1 and τ_2 that satisfy $\mathbf{f} = \int \boldsymbol{\mu} d\tau_i$, i.e., Bayes plausibility, their convex combination also satisfies Bayes plausibility. This together with

[10]Since θ is a discrete random variable with finitely many values, it is without loss of generality to focus on rating systems with finitely many signals.

[11]We formally show this in Lemma 2 in the Appendix.

the fact that $\int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau$ is linear in τ implies that $\mathcal{S}$ is convex. Now consider $\mathbf{q} \succcurlyeq_F \overline{\mathbf{q}}$. In order to show that $\overline{\mathbf{q}} \in \mathcal{S}$, we show that for every vector $\boldsymbol{\lambda} \in \mathbb{R}^N$, there exists $\mathbf{r} \in \mathcal{S}$ such that $\boldsymbol{\lambda}^T\mathbf{r} \geq \boldsymbol{\lambda}^T\overline{\mathbf{q}}$. Since this would also be true for $-\boldsymbol{\lambda}$, along any directions one can find two members of $\boldsymbol{\lambda}^T\mathcal{S}$ that are on either side of $\boldsymbol{\lambda}^T\overline{\mathbf{q}}$ on the real line. This observation together with the separating hyperplane theorem implies that $\overline{\mathbf{q}} \in \mathcal{S}$.

In order to show the inequality $\boldsymbol{\lambda}^T\mathbf{r} \geq \boldsymbol{\lambda}^T\overline{\mathbf{q}}$, we proceed by induction. When $N = 2$, the proof is straightforward. The definition (3) in this case implies that $\overline{q}_2 - \overline{q}_1 \leq q_2 - q_1$ (Figure 1). Since $\mathbf{f}^T\overline{\mathbf{q}} = \mathbf{f}^T\mathbf{q}$, then $\overline{\mathbf{q}}$ must lie on a line connecting $\mathbf{q}$ to $\left(\mathbf{f}^T\mathbf{q}\right)\mathbf{e}$, i.e., signaled qualities associated with no information. This implies our claim is true.

For $N > 2$, we show that the inequality either holds for full information or that it is possible to pull two consecutive states, focusing on rating systems that do not distinguish between states i and $i+1$, and reduce the number of states to $N-1$. We can then use the induction assumption, which proves our claim is true.

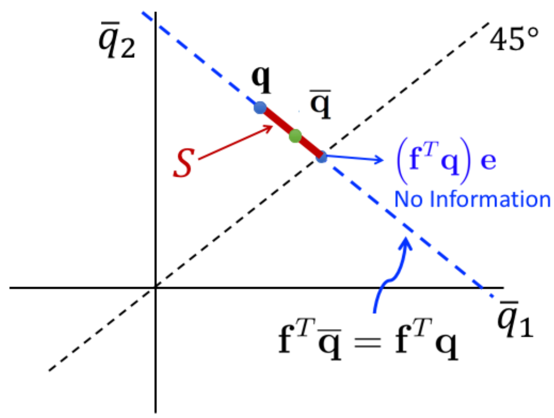

Figure 1: Depiction of $\overline{\mathbf{q}}$ satisfying (3) when $N = 2$.

It is worth comparing our main characterization result to those of other papers in the literature on Bayesian persuasion. A strand of that literature has considered a class of sender-receiver problems in which the receiver's action depends on her posterior expectations. For example, Gentzkow and Kamenica (2016), Kolotilin (2018), and Dworczak and Martini (2019) use a version of Blackwell's result (Blackwell (1953)[12]) and show that the posterior mean is a random variable that second-order stochastically dominates the state. They then use techniques from linear programming or optimization with stochastic dominance constraints to solve the problem. Our work is different from theirs in two respects: First, since we have to respect sellers' (i.e., senders') incentives, our variable of interest is the second-order expectation of the sender about the receiver's observed posterior. Second, we use

[12]A generalization of this is Strassen's theorem; see Theorem 7.A.1 in Shaked and Shanthikumar (2007)

a different formulation of the stochastic dominance relationship that is informative of the signal structure, as we illustrate below.

The above theorem can be used to provide a full characterization of the set of qualities and payoffs that arise from the equilibrium defined above, which is summarized in the following corollary:

Corollary 1. *The vectors of signaled qualities $\overline{\mathbf{q}}$ and qualities $\mathbf{q}$ arise in equilibrium if and only if the following are satisfied:*

$$\mathbf{q} \succcurlyeq_F \overline{\mathbf{q}}$$
$$\overline{q}_i - C\left(q_i, \theta_i\right) \geq \overline{q}_j - C\left(q_j, \theta_i\right), \forall i, j.$$

3.2 General Distribution of Seller Types

In this section, we extend our analysis to allow for the general distribution of types. Let $F(\theta)$ be a cumulative distribution function that has finitely many discontinuities. For any two increasing functions $\overline{q}(\theta)$ and $q(\theta)$ representing signaled and true quality, respectively, we say $q \succcurlyeq_F \overline{q}$ if the following holds:

$$\int_{\underline{\theta}}^{\theta} \overline{q}\left(\theta'\right) dF\left(\theta'\right) \geq \int_{\underline{\theta}}^{\theta} q\left(\theta'\right) dF\left(\theta'\right), \forall \theta \in \left[\underline{\theta}, \overline{\theta}\right] \tag{7}$$

$$\int_{\underline{\theta}}^{\overline{\theta}} \overline{q}\left(\theta\right) dF\left(\theta\right) = \int_{\underline{\theta}}^{\overline{\theta}} q\left(\theta\right) dF\left(\theta\right). \tag{8}$$

When $\overline{q}(\theta)$ and $q(\theta)$ are continuous, one implication of majorization is that for low values of θ, $\overline{q}(\theta) \geq q(\theta)$, while for higher values of θ, $q(\theta) \geq \overline{q}(\theta)$. Using this definition of F-majorization, we can show the following proposition:

Proposition 2. *Let $\overline{q}(\theta)$ and $q(\theta)$ be two functions representing signaled and true quality, respectively. Then, these functions arise from an equilibrium for some rating system if and only if they satisfy the following:*

1. *The profit function $\Pi(\theta) = \overline{q}(\theta) - C(q(\theta), \theta)$ is continuous for all θ. When its derivatives exist, it satisfies*
$$\Pi'(\theta) = -C_\theta\left(q(\theta), \theta\right). \tag{9}$$
2. *The functions $\overline{q}(\theta)$ and $q(\theta)$ are increasing in θ and satisfy $q \succcurlyeq_F \overline{q}$.*

We prove this proposition by considering the joint distribution of the posterior mean $\mathbb{E}[q|s]$ and $q(\theta)$. We approximate the distribution of $F(\cdot)$ with a sequence of discrete distributions whose supports are ordered according to the subset order, i.e., they are a filtration. We can apply the result of theorem 1 to construct an information structure associated with this discrete approximation. By compactness of the space of measures over the posterior mean and $q(\theta)$, these information structures must have a convergent subsequence with a limiting information structure. It thus remains to be shown that the expectation of the posterior mean conditional on $q(\theta)$ under this limiting information structure coincides with $\overline{q}(\theta)$. To show that, we resort to the martingale convergence theorem. In particular, given our construction of the discrete distributions, the support of such distributions can be used to construct a filtration. This filtration and the realization of $q(\theta)$ and posterior mean form a bounded martingale. As a result, we can apply Doob's martingale convergence theorem to show that the posterior mean conditional on $q(\theta)$ under the limiting information structure coincides with $\overline{q}(\theta)$. We formalize this argument in the appendix.

The conditions in Proposition 2 represent the incentive compatibility (9) and majorization. Incentive compatibility implies that the surplus function is continuous. However, it is possible that qualities (signaled and true) exhibit discontinuities. This implies that such discontinuities must occur at the same points and in such a way that $\Pi(\theta)$ is continuous.

3.3 Separating Rating Systems

While the above result fully characterizes the set of payoffs and allocations, it is not informative about the rating systems that implement a given pair of signaled and true quality functions, $\overline{q}(\theta)$ and $q(\theta)$. In this section, we describe what the majorization constraints imply about the rating systems that implement certain payoffs. While in general it is difficult to provide a full characterization of the rating systems that implement a certain pair of signaled and true quality, it is possible to provide a partial characterization of their properties.

Our first result describes when different sets of qualities must be separated by rating systems. We say a rating system (π, S) is *separating at* $\hat{q}$ if the set of signals generated by the types with $q(\theta) \leq \hat{q}$ is different from that generated by the types with $q(\theta) > \hat{q}$ almost surely. Formally, if we define the set of signals generated by the types below and above $\hat{\theta}$ as follows:

$$S_1 = \bigcup_{\theta: q(\theta) \leq \hat{q}} \text{Supp}\left(\pi\left(\cdot | q(\theta)\right)\right), S_2 = \bigcup_{\theta: q(\theta) > \hat{q}} \text{Supp}\left(\pi\left(\cdot | q(\theta)\right)\right),$$

then (π, S) is separating at $\hat{q}$ if

$$\int_{\Theta} \pi\left(S_1 \cap S_2 | q(\theta)\right) dF(\theta) = 0.$$

The following proposition states the condition under which a signal is separating at $\hat{q}$:

Proposition 3. *Let $\overline{q}(\theta)$ and $q(\theta)$ be a pair of signaled and true quality functions that satisfy the conditions in Proposition 2. Let (π, S) be a rating system for which $\overline{q}(\theta) = \mathbb{E}\left[\mathbb{E}\left[q|s\right] | q(\theta)\right]$. Then (π, S) is separating at $\hat{q}$ if and only if the majorization inequality (7) binds at $\hat{\theta} = \max_{\{\theta: q(\theta) = \hat{q}\}} \theta$.*

The proof is relegated to the appendix.

When (π, S) is separating at $\hat{q}$, then when receiving a signal that belongs to qualities below $\hat{q}$, a buyer is certain that the type she faces is below $\hat{\theta}$. Therefore, standard Bayes plausibility implies that the expectation of signaled qualities and true qualities conditional

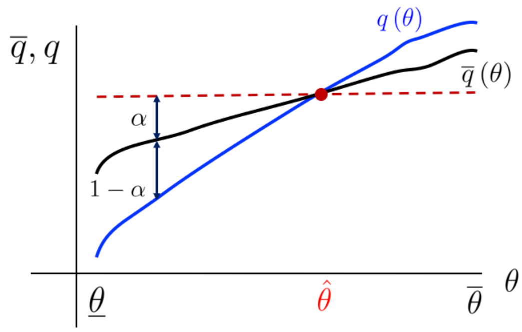

Figure 2: Full mixing ratings and their associated signaled quality

on types being below $\hat{\theta}$ must be equal. The reverse statement can be shown by considering the inequalities in the proof of Proposition 1. This implies that for a given pair of $\overline{q}(\cdot)$ and $q(\cdot)$ functions, a first step of identifying the rating system that delivers such qualities and signaled qualities is to identify the points at which the majorization constraint is binding.

The following illustrates one example where majorization is always slack.

Example. Full Mixing Rating Systems. Suppose that $\overline{q}(\theta)$ and $q(\theta)$ are both continuous, increasing, and satisfy $q \succcurlyeq_F \overline{q}$. Furthermore, suppose that there exist a unique $\hat{\theta}$ for which $\overline{q}\left(\hat{\theta}\right) = q\left(\hat{\theta}\right)$. This together with majorization implies that for values of $\theta < \hat{\theta}$, $\overline{q}(\theta) > q(\theta)$ holds, while for values of $\theta > \hat{\theta}$, $\overline{q}(\theta) < q(\theta)$ must hold. Moreover, it implies that the majorization inequality (7) never binds for values of $\theta < \overline{\theta}$. Let us define the function $\alpha(\theta)$ as follows:

$$\overline{q}(\theta) = \alpha(\theta)\, q(\theta) + (1 - \alpha(\theta))\, q\left(\hat{\theta}\right).$$

Given our assumption on $\overline{q}$ and q, $\alpha(\theta) \in [0, 1]$. This implies that a signal that reveals the quality with probability $\alpha(\theta)$ and reveals nothing (sends a generic signal) with probability $1 - \alpha(\theta)$ can implement the signaled quality function $\overline{q}(\theta)$. Figure 2 depicts the signaled qualities in this example. Since $\overline{q}(\cdot)$ and $q(\cdot)$ intersect only once, the value of α is between 0 and 1 and therefore well-defined. We refer to such rating systems as *full mixing*.

The above example illustrates the property that when the majorization constraint does not bind (in the interior of Θ) some pooling of signals is required; in this case, it is an extreme form as all types send the generic signal with a positive probability. As we will show, full mixing rating systems are a key feature of Pareto optimal rating systems.

Construction of Rating Systems While in general characterizing properties rating systems from a signaled qualities profile is difficult, we provide an algorithm to construct the rating system based on quality and signaled quality profiles – see Appendix . This algorithm generates a rating system when the type space is discrete. Moreover, it shows that a repeated application of a small class of rating systems – those that simply pool qualities together – can always implement a vector of signaled qualities.

4 Pareto-optimal Rating Systems

The results in the previous section provide a full characterization of the set of payoffs and qualities that can arise in equilibrium under an arbitrary rating system. In this section, we use this characterization to derive the properties of an optimal rating system given different objectives. Our focus will be on the set of Pareto-optimal allocations, i.e., those that maximize a weighted average of sellers' surplus. In Section 6, we consider alternative objectives, such as the revenue of the intermediary that charges a flat fee. Throughout the rest of our analysis, we assume that $F(\cdot)$ is continuous in θ.

The class of objectives that we consider is given by the following expression:

$$\int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta) \Pi(\theta) \, dF(\theta) \,. \tag{10}$$

In the above $\Pi(\theta)$ is the payoff of a seller of type θ, while $\lambda(\theta)$ is the welfare weight of sellers of different types; without loss of generality, we set $\int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta) \, dF = 1$. We consider a few cases for this objective in order to convey the main insight of optimal rating design: (1) total profits, i.e., $\lambda(\theta) = 1, \forall\theta$; (2) low-quality optimal, i.e., $\lambda(\theta)$ decreasing in θ; (3) high-quality optimal, i.e., $\lambda(\theta)$ strictly increasing in θ; and (4) mid-quality optimal, i.e., $\lambda(\theta)$ strictly increasing for $\theta < \theta^*$ and strictly decreasing for $\theta > \theta^*$.

Hence, the problem of Pareto-optimal rating design is to maximize the objective in (10) subject to incentive compatibility (as described in Proposition 2) and majorization constraints (7) and (8). This problem is not a convex programming problem, i.e., the constraint set is not convex mainly due to the presence of majorization and incentive constraints.[13] As a result, standard Lagrangian methods cannot be used. In what follows, we use perturbation arguments to prove our results.

4.1 Total Profits

In order to characterize optimal rating systems, it is useful to start from the first best benchmark, the one in which total profits is maximized. In the unconstrained optimum, ignoring the fact that θ is unobservable and that incentives are provided via rating systems, qualities must satisfy

$$C_q\left(q^{FB}(\theta), \theta\right) = 1.$$

[13]As shown by Guesnerie and Laffont (1984), it is possible to make assumptions about the cost function and transform variables in order to make the set implied by the incentive constraints convex. However, under such transformation, the majorization constraint becomes non-convex.

Letting signaled quality be defined by $\overline{q}(\theta) = q(\theta)$, it is straightforward to see that the pair $(\overline{q}, q)$ is incentive compatible and satisfies $q \succcurlyeq_F \overline{q}$. Moreover, since the majorization inequality (7) binds for all θ's, it implies that $\overline{q}$ should be implemented with an information system that fully reveals sellers' qualities. We use this simple benchmark as a point of comparison against other Pareto-optimal allocations.

4.2 Low-quality Optimal Ratings

To characterize low-quality-optimal rating systems, it is useful to consider the standard mechanism design problem in which there are no restrictions on $\overline{q}(\cdot)$: one in which $\overline{q}(\cdot)$'s are interpreted as monetary payments. In this case, the problem of solving for the optimal mechanism is similar to the familiar mechanism design problem such as that considered by Mussa and Rosen (1978) and Baron and Myerson (1982), among others. Thus, similar techniques can be used to solve this relaxed version of the problem.

This relaxed mechanism design problem is given by

$$\max \int \lambda(\theta)\, \Pi(\theta)\, dF(\theta) \tag{P}$$

subject to

$$\begin{aligned} \Pi(\theta) &\geq 0 \\ \Pi'(\theta) &= -C_\theta(q(\theta), \theta) \\ q(\theta) &: \text{increasing} \\ \int_{\underline{\theta}}^{\overline{\theta}} \Pi(\theta)\, dF(\theta) &= \int_{\underline{\theta}}^{\overline{\theta}} [q(\theta) - C(q(\theta), \theta)]\, dF(\theta). \end{aligned}$$

The solution can be found using standard techniques of solving mechanism design problems, as for example those described by Myerson (1981). In particular, there are two possibilities: (1) the cost function, $C(\cdot, \cdot)$, and distribution, $F(\cdot)$, are such that the monotonicity constraint on $q(\theta)$ is slack, and (2) the monotonicity constraint sometimes binds. In the first case, it is straightforward to show that $q(\theta)$ must satisfy

$$1 = C_q(q(\theta), \theta) - C_{\theta q}(q(\theta), \theta) \left[\frac{1 - F(\theta) - \int_\theta^{\overline{\theta}} \lambda(\theta')\, dF(\theta')}{f(\theta)} \right]. \tag{11}$$

Since $\lambda(\theta)$ is decreasing, the above implies that $C_q \leq 1$ since $C_{\theta q} \leq 0$. If $\overline{q}(\theta)$ and $q(\theta)$ are differentiable, we must have that

$$\overline{q}'(\theta) = C_q(q(\theta), \theta)\, q'(\theta) \leq q'(\theta).$$

Therefore, signaled qualities are flatter than actual qualities. The equality of the average value of signaled qualities and that of true qualities implies that the majorization inequality (7) is always satisfied. In other words, the optimal rating system is full mixing – see section

3.3. When the monotonicity constraint binds, Myerson (1981)'s ironing procedure can be used to find the optimal qualities. In that solution, either $q(\theta)$ is constant or it satisfies (11). Therefore, a similar argument can be used to show that the majorization inequality is always satisfied.

We thus have the following proposition:

Proposition 4. *A quality allocation $q(\theta)$ is low-quality-optimal if and only if it is a solution to the relaxed problem (P). Moreover, if the cost function $C(\cdot,\cdot)$ is strictly submodular, then a low-quality-optimal rating system is full mixing.*

The proof is relegated to the appendix.

Intuitively, in low-quality-optimal allocation, the goal is to reduce high-quality sellers' surplus as much as possible while at the same time respecting incentive compatibility. The existence of information rents arising from incentive compatibility implies that quality provision is always lower than that of the first best, i.e., $C_q \leq 1$ has to hold. Moreover, as (11) establishes, when $C(\cdot,\cdot)$ is strictly submodular, $C_q < 1$, so some degree of pooling is required in order to increase low-quality sellers' surplus.

4.3 High-quality Optimal Ratings

In this section, we discuss the properties of optimal rating systems when the Pareto weight of higher-quality sellers is higher. As in section 4.2, we can again consider the relaxed mechanism design problem without the majorization constraint. Since $\lambda(\theta)$ is increasing, the same argument as in section 4.2 shows that the solution to the relaxed problem (i.e., when the monotonicity constraint is slack) should satisfy $C_q > 1$. This, combined with the incentive constraint, implies that $\overline{q}(\cdot)$ will be steeper than $q(\theta)$ as a function of θ, and thus violates the majorization constraint. Intuitively, when $\lambda(\theta)$ is increasing, it is optimal to allocate profits to higher-quality sellers. However, since this is not incentive compatible, optimal allocations involve the overprovision of quality in order to make allocations incentive compatible, but this cannot be achieved, as our result in section 3 implies. Intuitively hiding any information will not help high-quality sellers as they will be mixed with low-quality sellers and thus get lower prices in equilibrium. Furthermore, given that all information is revealed, first best allocations are optimal.

The following proposition establishes that high-quality-seller–optimal rating systems must be fully revealing:

Proposition 5. *A quality allocation is high-quality optimal if and only if it satisfies $q(\theta) = q^{FB}(\theta)$. Moreover, a high-quality optimal rating system is fully revealing.*

For a fully revealing rating system, the majorization constraint always binds. The proof of this proposition involves coming up with perturbations of an allocation for which majorization is slack and therefore reaching a contradiction with the optimality.

In particular, in order to prove Proposition 5, we focus on a relaxed version of the problem, where we only impose incentive compatibility as

$$\Pi(\theta) - \Pi(\underline{\theta}) \leq -\int_{\underline{\theta}}^{\theta} C_\theta(q(\theta'), \theta') \, d\theta'. \tag{12}$$

When the above inequality binds for all values of θ, it becomes equivalent to the incentive compatibility constraint. [14]We show that the solution of this relaxed problem is the first best allocation and thus at the solution this inequality binds.

Suppose that the majorization constraint is slack on an interval. Then, two properties have to be true for any such interval:

1. It must be that over this interval $C_q(q(\theta), \theta) \geq 1$. To see that, suppose to the contrary that for a subinterval I, $C_q(q(\theta), \theta) < 1$. Then, one can consider a perturbation of $q(\theta)$ over such a subinterval that increases $q(\theta)$, given by $\delta q(\theta)$. Additionally, we perturb $\overline{q}(\cdot)$ as follows:

$$\delta\overline{q}(\theta) = \begin{cases} C_q(q(\theta), \theta)\, \delta q(\theta) + \int_I [1 - C_q(q(\theta), \theta)]\, \delta q(\theta)\, dF(\theta) & \theta \in I \\ \int_I [1 - C_q(q(\theta), \theta)]\, \delta q(\theta)\, dF(\theta). & \theta \notin I \end{cases}$$

   This perturbation increases $q(\theta)$ for the θ's in I, compensates these types for their cost increase, and since $C_q < 1$, allocates the extra surplus generated by this perturbation across all types. Under this perturbation, $\delta\Pi(\theta) = \delta\Pi(\underline{\theta})$ for all values of θ (Figure 3).

   The perturbation increases some values of $q(\theta)$ and leaves the rest unchanged, and $C_{\theta q} \leq 0$, so it does not violate (12). As majorization is slack over this interval, it is always possible to make $\delta q(\theta)$ small enough so that it holds under the perturbed allocation. Therefore, this perturbation increases profits and satisfies all the constraints.

2. Over this interval, the incentive constraint (12) is binding. If not, it is possible to take subinterval I, increase $\Pi(\theta)$ for high values of θ, and decrease $\Pi(\theta)$ for low values of θ without violating the incentive constraint. This is possible because the majorization inequalities are slack over I. Since $\lambda(\theta)$'s are increasing, this only increases the objective. Thus, at the optimum, the incentive constraint is binding.

It is fairly straightforward to show that the above properties are in contradiction with a slack majorization constraint. In particular, since over this interval $C_q \geq 1$ and the incentive constraint is binding, then it must be that $\overline{q}(\cdot)$ is steeper than $q(\cdot)$. Hence, majorization will be violated and the claim is proven. We formalize this argument in the appendix.

[14]Note that the inequality in (12) is derived from integrating the envelope condition when sellers are restricted to lie upward, i.e., when they can only pretend to be of a higher type. It is thus similar to restricting attention to upward incentive constraints in an environment with discrete types. That these constraints are the relevant ones here is natural since the objective is to allocate profits to higher-quality sellers.

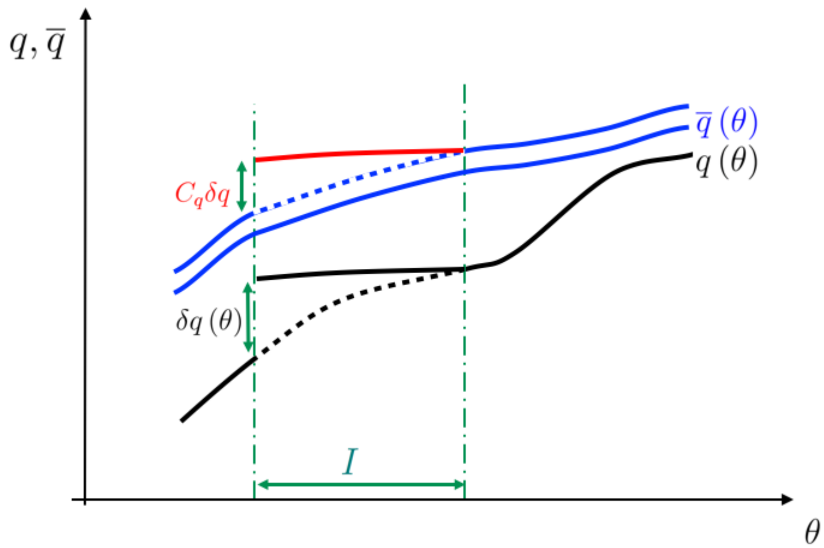

Figure 3: Perturbation of qualities and signaled qualities when $C_q(\theta) < 1$ and majorization is slack. The graph of $\overline{q}(\cdot)$ is shifted up by $\int_I (1 - C_q)\,\delta q dF > 0$.

The above result points toward a force opposite to that discussed in section 4.2. That is, a rating system is unable to reallocate profits to higher-quality sellers. In particular, when the objective calls for reallocating profits to higher-quality sellers, the best that can be done is to reveal all information. This would imply that the quality allocation coincides with the first best. In the next section, we show that this insight holds also for other objectives.

4.4 Mid-quality Optimal Ratings

In this section, we illustrate the insight from section 4.3 that when profits must be allocated to sellers with higher quality – in this case mid-quality sellers – then optimal rating systems must involve revealing information. In particular, we assume that there exists θ^* for which $\lambda(\theta)$ is strictly increasing in θ when $\theta \leq \theta^*$ and $\lambda(\theta)$ is strictly decreasing in θ when $\theta \geq \theta^*$.

The following proposition characterizes optimal rating systems:

Proposition 6. *Suppose that $\lambda(\theta)$ is hump-shaped. Then there exists $\hat{\theta} \leq \tilde{\theta} < \theta^*$ such that for all values of $\theta \leq \hat{\theta}$, $q(\theta) = q^{FB}(\theta)$; for all values of $\theta \in \left[\hat{\theta}, \tilde{\theta}\right)$, $q(\theta) = q^{FB}\left(\hat{\theta}\right)$; while it is full mixing for values of q above $q\left(\tilde{\theta}\right)$.*

Figure 4 depicts the structure of the optimal rating system described in proposition 6. As it can be seen, when the objective values the profits of medium-quality sellers, it is always optimal to hide some information about high-quality sellers. In particular, by creating uncertainty about middle- and high-quality types, the rating system can deliver higher profits to mid-quality types. For low-quality sellers, as in the case discussed in section 4.3, the rating system is fully revealing. Additionally, a possible element of mid-quality optimal allocations is bunching of types. This occurs mainly because at $\tilde{\theta}$, the optimal level of profit could be less than the first best – since this allows the allocation to push profits to higher quality sellers. This feature combined with the fact that majorization is binding below $\tilde{\theta}$ implies that there must be bunching of types.

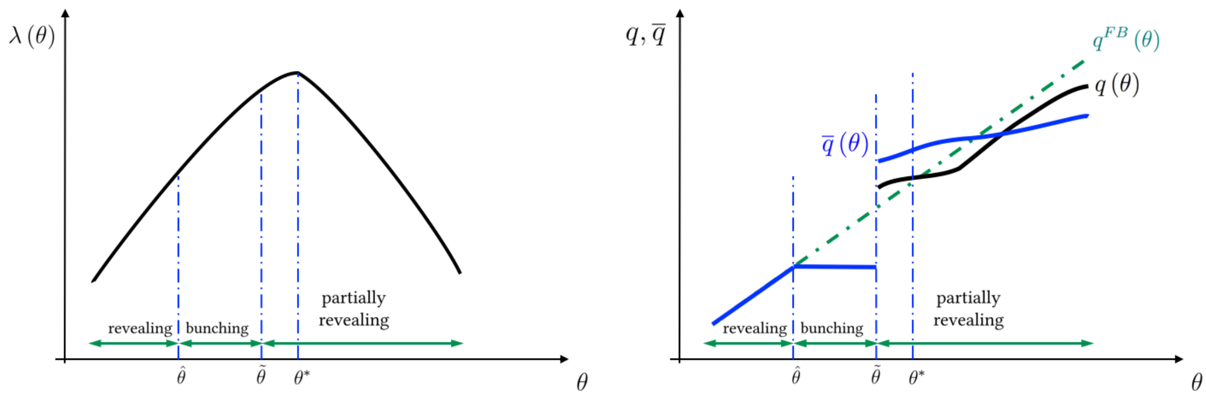

Figure 4: Mid-quality optimal ratings. The left panel depicts the position of the full revelation and bunching cutoff; the right panel depicts optimal qualities and signaled qualities.

The resulting rating system reveals all information for low qualities; reveals only that $q \in \left[q\left(\hat{\theta}\right), q\left(\tilde{\theta}+\right)\right)$, and is a full mixing rating system when $q \geq q\left(\tilde{\theta}+\right)$ – $q(\theta+)$ is the right limit of $q(\theta')$ as θ' approaches θ.

Our proof of Proposition 6 involves elements that are similar to the perturbation argument in the proof of Proposition 5. In particular, we first focus on a more relaxed version of the problem, one in which for values of $\theta \leq \theta^*$, the inequality (12) must be satisfied, while for values of $\theta \geq \theta^*$, the inequality

$$\Pi\left(\overline{\theta}\right) - \Pi(\theta) \geq -\int_{\theta}^{\overline{\theta}} C_{\theta}\left(q(\theta'), \theta'\right) d\theta' \tag{13}$$

must be satisfied; we will then show that both sets of inequality must bind at the optimum and thus the solution of the relaxed problem coincides with that of the main problem. However, our perturbations allow the inequalities to be relaxed. [15]

In this relaxed version of the problem, we are able to show that if the majorization constraint binds for some $\theta' < \theta^*$, then it must bind for all values of $\theta \leq \theta'$. To do this, we use the same perturbations discussed in section 4.3.

[15] Similar to inequality (12), (13) is derived from integrating the envelope condition between θ and $\overline{\theta}$ when sellers can only lie downward.

For values of $\theta > \theta^*$, we can show that the majorization constraint never binds. We demonstrate that for values of $\theta > \theta^*$, $C_q(q(\theta), \theta) \leq 1$. Otherwise, a perturbation that reduces quality increases the objective and keeps majorization satisfied. Moreover, for all values of $\theta > \theta^*$, (13) must bind, which implies that $\Pi(\theta)$ must be continuous. We then show that if majorization is to bind at some $\overline{\theta} > \theta' > \theta^*$, then $C_q(q(\theta'), \theta') > 1$, which is a contradiction.

An interesting observation is that even though the objective puts the highest weight on sellers of type θ^*, the rating system combines the types below θ^* with those above it, i.e., $\tilde{\theta} < \theta^*$, because of the incentive constraint. In particular, if one solves for the optimal rating system when θ is restricted to be in $\left[\theta^*, \overline{\theta}\right]$, at the optimum $q(\theta^*) = q^{FB}(\theta^*)$ while $\overline{q}(\theta^*) > q(\theta^*)$. In other words, the surplus generated by type θ^* is higher than in the first best. As a result, continuity of $\Pi(\theta)$ cannot hold since its value is at most equal to the first best.

Finally, an optimal rating system must necessarily create a jump in qualities. That is, it should be designed in a way that no one chooses qualities in $\left[q\left(\tilde{\theta}-\right), q\left(\tilde{\theta}\right)\right]$. This is because for sellers below $\tilde{\theta}$ the allocations are fully revealing, while the rating system partially pools sellers above $\tilde{\theta}$ with higher types. For payoffs to be continuous, as implied by the incentive compatibility, allocations must exhibit a jump.

In summary, our results in this section highlight the main trade-offs in reallocating profits using rating systems. On the one hand, when profits should be allocated from lower-quality to higher-quality types of sellers, the rating system must reveal all information. On the other hand, when the opposite should occur, the rating system must be partially revealing. One can use this as a general guide in designing a rating system.

We should note that while the immediate interpretation of the analysis here is that of characterizing the Pareto frontier of this environment, one can interpret the Pareto weights as arising from heterogeneous outside options – see Jullien (2000) for an analysis of this problem – where $\lambda(\theta)$'s are associated with tightness of the participation constraints.[16] While equivalent to our problem, the problem of analyzing the solution with heterogeneous outside options is much less tractable. While it is not possible to show, one can associate higher outside option loosely with higher welfare weights. Under this admittedly loose interpretation, our analysis provides a connection between design of rating systems and the side of the market where entry conditions are tighter, i.e., what is sometimes referred to as shorter side of the market. According to our analysis, when high quality sellers are the shorter side of the market, then ratings should reveal all information in order to relax their entry. If this occurs for mid- or low-quality sellers, optimal rating systems must involve some mixing.

[16] In Dworczak et al. (forthcoming), agents are heterogeneous in their marginal value of money and the λ's represent this heterogeneity.

5 Optimal Ratings with Random Quality Outcomes

The model used in the previous section assumes that sellers can precisely choose their level of quality. However, often, sellers may not be able to make such a precise choice. For example, two sided platforms often use measures of quality that are not fully controlled by the service provider. These platforms often rely on buyers' feedback which could be subject to randomness: an eBay seller can be subject to shipment delays that are not fully controlled by her; an Airbnb host can be matched with an overcritical guest, etc. In this section, we allow for random quality outcomes. As we will show, while the optimal rating systems are deterministic partitions, they share some similarities to those derived in section 4.

Consider the model in section 2 and suppose that when a seller chooses q, the realized quality is given by x, where x is random. Without loss of generality, assume that $q, x \in [0, 1]$. We denote the distribution (p.d.f) of x conditional on choice of q by $g(x|q)$ while its cumulative distribution is $G(x|q)$. As in 2, the sellers can be of different types θ and the cost of choosing q is given by $C(q, \theta)$ which satisfies Assumption 1. We make the following assumptions on $g(x|q)$:

Assumption 2. *The distribution function $g(x|q)$ satisfies:*

1. *Average value of x is q, i.e., $\int_0^1 xg(x|q)\,dx = q$.*

2. *The distribution function $g(x|q)$ is continuously differentiable with respect to x and q for all values of $x \in [0, 1]$ and $q \in (0, 1)$.*

3. *The distribution function $g(x|q)$ satisfies full support, i.e., $g(x|q) > 0, \forall x \in (0, 1)$ and monotone likelihood ratio, i.e., $g_q(x|q)/q(x|q)$ is strictly increasing in x.*

The first part of this assumption is a normalization of the mean of x.[17] The second part is a differentiability assumption and is made so that we can use standard results from calculus of variation. The third part of the assumption implies that one cannot back out the chosen q by an observation of x and that the likelihood ratio g_q/g is increasing. The latter implies that an increase in q shifts the distribution of x to the right, i.e., increases the distribution of x in the sense of first-order stochastic dominance.

As in section 2, we assume that the intermediary observes the quality experienced by the buyers x and designs a rating system represented by (S, π) where $\pi(s|x)$ is the distribution of signals conditional on the realization of quality x. As before, prices are given by buyers' posterior beliefs and expected prices for each realization of quality x are given by

$$\overline{x}(x) = \int \mathbb{E}[x|s]\,\pi(ds|x).$$

[17]More generally one can think of an action $a \in \mathbb{R}$ that controls the distribution of quality outcomes. As long as the average value of the outcome is a concave and increasing function of a, the normalization in part 1 of Assumption 2 is without loss of generality.

We impose that $\overline{x}(x)$ must be increasing in x.[18] Recall that, in the deterministic case, incentive compatibility implies the monotonicity of the signaled qualities. This assumption is also a realistic one as leading examples of rating and certification typically exhibit monotone signals.[19]

Similar to the model with deterministic qualities, a characterization result as in Theorem 1 and Proposition 2 should hold here. However, the definition of majorization now depends on the distribution of x. For an average quality profile $\{q(\theta)\}_{\theta\in\Theta}$, the (cumulative) distribution of x is given by

$$H(x) = \int_{\Theta}\int_0^x g(x'|q(\theta))\, dx' dF(\theta)$$

We can, thus, say that $\overline{x} \preccurlyeq_H x$ if and only if it satisfies $\int_0^x [\overline{x}(x') - x']\, dH(x') \geq 0$ and $\int_0^1 [\overline{x}(x) - x]\, dH(x) = 0$.

Given a rating system and the signaled qualities it induces, i.e., the function $\overline{x}(x)$, sellers choose q optimally and the choice of average quality by the seller of type θ, $q(\theta)$ must satisfy the following incentive compatibility

$$q(\theta) \in \arg\max_{q\in[0,1]} \int_0^1 \overline{x}(x)\, g(x|q)\, dx - C(q,\theta) \tag{14}$$

In order to simplify the problem of optimal rating design, we replace the above incentive constraint with its local version

$$C_q(q(\theta), \theta) = \int_0^1 \overline{x}(x)\, g_q(x|q(\theta))\, dx \tag{15}$$

The above constraint replaces incentive compatibility with its first order condition. When this constraint is sufficient, we say first-order approach is valid. We derive all of our results under the assumption that first-order approach is valid.

Given the formulation of majorization and incentive compatibility, the problem of finding a Pareto optimal rating system is then to choose $q(\theta)$ and $\overline{x}(\cdot)$ to maximize a weighted average of sellers' payoffs $\int_{\Theta} \lambda(\theta)\, \Pi(\theta)\, dF(\theta)$ subject to the local incentive constraint 15 and $x \succcurlyeq_H \overline{x}$.

5.1 Optimality of Monotone Partitions

In this subsection, we show that optimal signal when the quality provision is random can be obtained within monotone partitions in which in each partition, we either fully reveal all information about x or we just inform the buyer that the sellers' quality is within the partition.

[18] This is partly because we do not know if our characterization result in section 3 applies absent this monotonicity assumption.

[19] The incentive compatibility constraint (14) implies that $\int_0^1 \overline{x}(x)\, g(x|q(\theta))\, dx$ and $q(\theta)$ must be increasing in θ. Under Assumption 2, this is satisfied when $\overline{x}(\cdot)$ is monotone increasing.

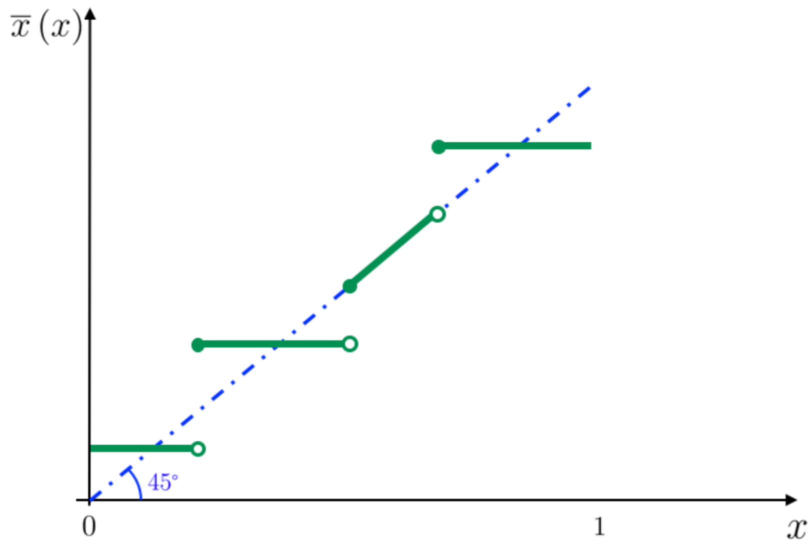

Figure 5: Signaled qualities for a monotone partition rating system.

Let us formally define our notion of deterministic monotone partitions. A rating system (S, π) is called a *monotone partition* if there exists a partition of $[0, 1]$ to a collection of sets $\{I_\alpha\}_{\alpha \in A}$ where: 1. each I_α is either a closed or half-open interval of the form $[x_1, x_2)$, 2. for all $\alpha, \beta \in A$, I_α and I_β are ranked, i.e., either $\min I_\alpha \geq \sup I_\beta$ or $\min I_\beta \geq \sup I_\alpha$, 3. For each $\alpha \in A$ there exists a unique signal $s_\alpha \in S$ such that $\pi(\{s_\alpha\} | x) = 1$ for all $x \in I_\alpha$. In words, a monotone partition either fully reveals each value of quality (in which case I_α is a singleton) or pools it with an interval around it. Note that the signaled qualities $\overline{x}(x)$ associated with a monotone partition rating system is always of the form depicted in Figure 5. The points at which the majorization constraints are binding create the partition I_α wherein between any two such points $\overline{x}(x)$ is constant and equal to the mean value of x conditional on x belonging to such an interval.[20]

The following proposition establishes that all optimal rating systems must be monotone partitions:

Proposition 7. *Suppose that the first order approach is valid and Assumption 2 holds. Then a Pareto optimal rating system is a monotone partition.*

Proof. We show the claim by first showing that for all x, either the monotonicity constraint is binding or the majorization constraint is binding at the optimum. Suppose to the contrary that this does not hold. Note that a change in $\overline{x}(x)$ for a measure zero of x's, does not affect the objective, and the majorization constraint. This implies that in order to achieve a contradiction, we need to rule out an interval in which neither majorization nor monotonicity

[20]The assumption that I_α's are half-open intervals implies that $\overline{x}(\cdot)$ is right continuous. This is without loss of generality as the distribution of x does not have atoms.

constraint is binding. Suppose that there exists an interval $I = [x_1, x_2]$ for which majorization and monotonicity are slack. Note that under the validity of the first order approach, the optimal rating system must be a solution to the following planning problem:

$$\max_{q(\cdot),\overline{x}(\cdot)} \int_{\Theta} \lambda(\theta) \left[\int_0^1 \overline{x}(x) g(x|q(\theta)) dx - C(q(\theta), \theta) \right] dF(\theta) \tag{P1}$$

subject to

$$\int_0^1 \overline{x}(x) g_q(x|q(\theta)) dx = C_q(q(\theta), \theta)$$

$$\int_0^x [\overline{x}(x') - x'] \int_{\Theta} g(x'|q(\theta)) dF(\theta) dx' \geq 0$$

$$\int_0^1 [\overline{x}(x) - x] \int_{\Theta} g(x|q(\theta)) dF(\theta) dx = 0$$

$$\overline{x}(x) - \overline{x}(x') \geq 0, \forall x \geq x'$$

By combining the Theorems 1 in section 9.3 and 9.4 of Luenberger (1997), there must exist lagrange multipliers $\gamma(\theta)$ – for the incentive compatibility constraint, and a positive lagrange multiplier $M(x)$ associated with the majorization constraint and $m-$ the lagrange multiplier associated with the last constraint together with $\eta(x, x') \geq 0$ for the monotonicity constraint.

The optimality condition for values of $x \in I$ are given by

$$\int_{\Theta} \lambda(\theta) g(x|q(\theta)) dF(\theta) + \int_{\Theta} \gamma(\theta) g_q(x|q(\theta)) d\theta$$
$$+ \int_x^1 M(x') dx' \int_{\Theta} g(x|q(\theta)) dF(\theta)$$
$$+ m \int_{\Theta} g(x|q(\theta)) dF(\theta) = 0 \tag{16}$$

Moreover, since by assumption the majorization constraint is slack for $x \in I$, $\mu(x) = \int_x^1 M(x') dx'$ is constant for all $x \in I$. We can write the above as

$$\int_{\Theta} (\lambda(\theta) + \mu(x_1) + m) g(x|q(\theta)) dF(\theta) + \int_{\Theta} \gamma(\theta) g_q(x|q(\theta)) dx = 0, \forall x \in I \tag{17}$$

Let $\hat{x}(x)$ be defined as

$$\hat{x}(x) = \begin{cases} \overline{x}(x) & x \notin I \\ \frac{\int_I \overline{x}(x) dH(x)}{\int_I dH(x)} & x \in I \end{cases}$$

Note that $\overline{x} \succcurlyeq_H \hat{x}$ and therefore, by transitivity of majorization $x \succcurlyeq_H \hat{x}$. Since $\hat{x}(\cdot)$ and $\overline{x}(\cdot)$ only differ on I and (17) holds, the value of lagrangian is the same for $\hat{x}$ and $\overline{x}$. Thus, we can replace $\overline{x}$ with a signaled quality function for which monotincity is binding for I.

Finally, note that since $\overline{x}(\cdot)$ is a bounded function, $\int_0^x [\overline{x}(x') - x'] \int_\Theta g(x'|q(\theta)) dx' dF(\theta)$ is a continuous function. Hence, the points at which it takes a positive value is an open subset of $[0,1]$. Thus, it is union of disjoint intervals. For values of x belonging to any of the intervals, the monotonicity constraint must be binding – by the previous step. Hence, $\overline{x}(\cdot)$ is constant for any interval where majorization is slack. Additionally, the points at which the majorization constraint is binding is a closed subset of $[0,1]$ and thus it is a union of disjoint closed intervals (possibility isolated points). Over any such interval, taking derivative of the binding majorization constraint with respect to x implies that $\overline{x}(x) = x$. This establishes the claim. □

The above proposition implies that when quality outcomes are random, one does not need rating systems with random signals – as in section 4 – to provide incentives. Unlike in the deterministic model of section 4, deterministic monotone partitions do not lead to bunching of types. This is due to the fact that the realization of quality is random and cannot be fully controlled by the sellers. For example, in the deterministic model, a montone partition as shown in Figure 5 necessarily leads to bunching of types at points of discontinuity of $\overline{x}(\cdot)$ while this is not the case when x is random.

5.2 An Auxiliary Problem and a Mathematical Result

Before characterization of optimal rating systems, we first provide a mathematical result regarding optimal signaled qualities in a class of auxiliary problems that helps us in characterization of optimal rating systems. For an arbitrary function $\Gamma(x)$, consider the following optimization problem

$$\max_{\overline{x}(\cdot)} \int_0^1 \Gamma(x) \overline{x}(x) h(x) dx \tag{P'}$$

subject to

$$\int_0^x [\overline{x}(x') - x'] h(x') dx' \geq 0$$

$$\int_0^1 [\overline{x}(x) - x] h(x) dx = 0$$

$$\overline{x}(x) \geq \overline{x}(x'), \forall x \geq x'.$$

Note that for any solution of (P1), the problem of choosing $\overline{x}(\cdot)$ conditional on the choice of $q(\theta)$ boils down to solving a problem of the form (P') where $\Gamma(x)$ is a function of Pareto weights $\lambda(\cdot)$, distribution functions $g(x|q(\theta))$, and lagrangian multipliers associated with the incentive constraints (15). We will refer to the function $\Gamma(x)$ as the *gain function*. As we show in the following proposition, the main determinant of the solution of (P') is the derivative of $\Gamma'(\cdot)$ and its sign. We refer to a solution as an *alternating partition*, if $[0,1]$ can be partitioned to a collection of intervals and each interval is either fully pooled or fully revealed with no two consecutive intervals being of the same type. The following proposition, shows that under some fairly general assumptions on the gain function, the optimal solution of problem (P') is an alternating partition:

Proposition 8. *Suppose that a gain function $\Gamma(x)$ is continuously differentiable and that its derivative changes sign $k < \infty$ times, i.e., we can partition $[0,1]$ into k intervals where in each interval $\Gamma'(x)$ has the same sign but not in two consecutive intervals. Then, the optimal information structure is an alternating partition with at most k intervals.*

The proof is relegated to the appendix.

The above proposition provides a guideline for finding a solution to (P'). It generalizes the intuition that if $\Gamma(x)$ is increasing, then the solution of (P') is fully revealing while if $\Gamma(x)$ is decreasing, the solution is pooling. The main idea of the proof is due to the following insights which come from examining the optimality condition (16): 1. when the majorization constraint binds over an interval, the gain function must be strictly increasing; 2. if $[x_1, x_2)$ is a pooling interval, then $\Gamma(x_1) \geq \Gamma(x_2)$. Using these insights, we show that every local optimum and its subsequent local minimum must belong to the same pooling interval.

5.3 A Two-Type Example

While the above results illustrate the sufficiency of monotone partitions and a general understanding of optimal ratings, the exact nature of optimal rating systems depend on the details of the distribution of qualities and welfare weights. In what follows, we use a two-type example to illustrate that some of the insights from the deterministic case carry through and shed light on the determinants of optimal rating design in the presence of random quality outcomes.

Specifically, suppose that $\Theta = \{\theta_1, \theta_2\}$ with $\theta_1 < \theta_2$, $f(\theta_j) = f_j$, and $\lambda(\theta_1) = 1, \lambda(\theta_2) = 0$.[21] Before stating our formal result, we provide a heuristic analysis of the main determinants of optimal rating. Suppose that the lagrange multipliers on the incentive compatibility constraints are γ_j . Then, the *gain* function associated with optimal rating design is given by:

$$\Gamma(x) = \frac{g(x|q_1)}{h(x)} \left(1 + \gamma_1 \frac{g_q(x|q_1)}{g(x|q_1)} + \gamma_2 \frac{g_q(x|q_2)}{g(x|q_2)} \frac{g(x|q_2)}{g(x|q_1)}\right), \tag{18}$$

where in the above $h(x) = f_1 g(x|q_1) + f_2 g(x|q_2)$. Formally, given q_1 and q_2, optimal $\overline{x}(x)$ must maximize $\int_0^1 \Gamma(x)\overline{x}(x)h(x)dx$ subject to monotonicity and majorization.

Analyzing the terms in the gain function identifies two forces that shape the properties of the optimal rating system:

1. Redistributive: The first term in the gain function $g(x|q_1)/h(x)$ is a decreasing function of the likelihood ratio $g(x|q_2)/g(x|q_1)$. Under part three of Assumption 2, i.e., the monotone likelihood ratio property, this likelihood function is increasing in x and as a result the term $g(x|q_1)/h(x)$ is decreasing in x. Thus, when $\gamma_1 = \gamma_2 = 0$, i.e., when we do not have to worry about the effect of the rating system on incentives,

[21] As we show in section **??**, maximizing revenue from a flat fee for an intermediary leads to the same outcome as this particular Pareto optimal rating.

then optimal rating system is simply one that provides no information. This is because in this case, the gain function is positive for low values of x and negative for high values.

2. Incentive: The second and third term in the gain function represents the importance of incentive provision for types 1 and 2. The function $g_q(x|q)/g(x|q)$ is an increasing function; negative for low values of x and positive for higher values of x. As a result, the second term creates a force for information revelation. In fact, when γ_1 and γ_2 are very large, the gain function $\Gamma(x)$ becomes increasing and as a result it is optimal to reveal all information.

At the optimum, the exact nature of the optimal rating system depends on how these forces interact. While the guiding principle for the design of rating systems is Proposition 8, in what follows, we provide conditions for which revelation must occur for high and low values. As we show later, various classes of distribution functions $g(x|q)$ satisfy this assumption:

Assumption 3. *For arbitrary $q_2 > q_1$, define the function $\hat{x}(z)$ as the solution of $z = g(\hat{x}(z)|q_2)/g(\hat{x}(z)|q_1)$. The function $\hat{x}(z)$ must satisfy the following properties:*

1. *The function $\phi(z) = g_q(\hat{x}(z)|q)/g(\hat{x}(z)|q)$ satisfies $\phi''(z) \leq 0$,*
2. *The function $\psi(z) = zg_q(\hat{x}(z)|q)/g(\hat{x}(z)|q)$ satisfies $\psi''(z) \geq 0$,*
3. *The function $\phi''(z)/\psi''(z)$ is increasing in z.*

Using the above assumption, we have the following proposition:

Proposition 9. *Suppose that Assumptions 2 and 3 hold. Furthermore, suppose that $\Theta = \{\theta_1, \theta_2\}$ with $\theta_1 < \theta_2$ and that $\lambda(\theta_1) = 1, \lambda(\theta_2) = 0$. If at the optimum $q_2 \geq q_1$, then there exists two thresholds $x_1 < x_2$ where optimal rating system is fully revealing for values of x below x_1 and above x_2 while it is pooling for values of $x \in (x_1, x_2)$.*

Proof is relegated to the Appendix.

Under Assumption 3, the incentive effects are strongest for extreme values of x while the the redistributive force is strongest for mid values of x. As the Proposition illustrates optimal rating system pools intermediate values of x while fully reveal extreme values. Roughly speaking the full revelation of extreme values of x are associated with incentive provision for types 1 and 2. Under Assumption 3, the incentive effect for type 1 – the second term in the gain function (18) – is steepest for low values of x while the incentive effect for type 2 – the third term in the gain function (18) – is steepest for high values of x. As a result, mid-values of x are pooled, i.e., redistributive effect dominates, while at the extremes incentive effects dominate.

Some examples of distributions that satisfy Assumption 3 are:

1. P.d.f. is a power of x: $G(x|q) = x^{\frac{q}{1-q}}$ which implies that $g_q(x|q)/g(x|q) = \frac{1}{(1-q)^2}\left(\log x + \frac{1-q}{q}\right)$.

2. P.d.f. is a power of $1-x$: $G(x|q) = 1-(1-x)^{\frac{1-q}{q}}$ which implies that $g_q(x|q)/g(x|q) = \frac{1}{q^2}\left(-\frac{q}{1-q} - \log(1-x)\right)$.

3. P.d.f. is exponential of x : $G(x|q) = \left(e^{\lambda(q)x} - 1\right)/\left(e^{\lambda(q)} - 1\right)$ for some function $\lambda(q)$. This implies that $g_q/g = A(q)\log x + B(q)$. A similar property holds for $G(x|q) = \left(1 - e^{-\lambda(q)x}\right)/\left(1 - e^{-\lambda(q)}\right)$.

The analysis in this section illustrates two main lessons: 1. montone and alternating partitions are optimal when qualities are random, 2. the interplay between redistributive and incentive effects determine when outcomes are pooled and when fully revealed. While the details of the distribution function $g(x|q)$ matters, our analysis suggests that full revelation is important for the extreme realizations.

6 Extensions

In this section, we show how our results and analysis would extend beyond the model considered above.

6.1 Revenue-maximizing Intermediary

In the above analysis, we have considered optimal rating systems under Pareto optimality. However, it is often the case that intermediaries are self-interested. Here, we discuss the incentives of a revenue-maximizing intermediary that can charge the sellers a flat fee for entry.

Since the intermediary is a monopolist, a flat fee charged to all sellers that enter the market might lead to exclusion of some sellers. In other words, if the intermediary charges a fee e, sellers only enter if their payoff is higher than e. Note that since profits of the sellers depend on the rating system, $\pi(s|q)$, their decision to enter depends on the rating system. Thus, for any e and rating system $\pi(\cdot)$, there must exists an entry cutoff for sellers' types given by $\hat{\theta}(e, \pi(\cdot))$ which satisfies

$$e = \Pi\left(\hat{\theta}\right) = \max_{q'} \int_S \mathbb{E}[q|s]\,\pi(ds, q') - C\left(q', \hat{\theta}\right). \tag{19}$$

Note that when the right hand side of (19) is higher than e for all values of θ, then $\hat{\theta} = \underline{\theta}$.[22] Note that given the entry cutoff, the revenue of the intermediary is given by $e\left[1 - F\left(\hat{\theta}(e, \pi(\cdot))\right)\right]$. The problem of the intermediary is thus to choose an entry fee and rating system to maximize this revenue.

An insight that helps us characterize the optimal rating system from the intermediary's perspective is that we can think about the intermediary choosing the cutoff $\hat{\theta}$ and the rating system $\pi(\cdot)$ and using (19) to calculate the required fee that induces the entry of types of

[22] We have assumed deterministic quality here. The analysis does not really change when quality is random.

$\hat{\theta}$ and higher. Viewing the intermediary's problem this way, we can write its revenue as $\Pi\left(\hat{\theta}\right)\left[1-F\left(\hat{\theta}\right)\right]$. Thus, given a $\hat{\theta}$, the problem of finding the optimal rating system is to maximize the payoff of type $\hat{\theta}$, $\Pi\left(\hat{\theta}\right)$. This is the same as the problem studied in section (4.2). We thus have the following Proposition:

Proposition 10. *A revenue-maximizing optimal rating system is full mixing.*

Note that the introduction of the entry fee charged by the intermediary creates distortions by limiting entry and thus reduces total surplus. Nevertheless, conditional on entry the allocation is efficient – in the sense discussed in section 4. That is, given the set of active types, optimal rating system is a low-quality optimal rating system with all the weight on the lowest type that enters.

In analyzing the problem of the intermediary, it is also possible to consider heterogeneity in outside options for sellers. In particular, sellers' outside option can be dependent on their type, θ. In this case, the problem of the intermediary becomes similar to the objectives considered in section 4 where one can interpret the welfare weights $\lambda(\theta)$ as the Lagrange multipliers on the participation constraints faced by the intermediary. While in general solving the resulting mechanism design problem is difficult,[23] one can loosely argue that these multipliers are positively associated with the outside options, i.e., higher outside options are associated with higher multipliers. Thus, under this interpretation, our analysis implies that when higher quality sellers have a tighter participation constraint, then optimal ratings should be perfectly revealing. As Hui et al. (2020) have illustrated using a change in eBay's certification policy, middle-quality sellers' entry decision is more sensitive to changes in information policy. This evidence suggests that the type of objectives in section 4.4 is more relevant in that context.

6.2 The Role of Entry

In our analysis so far, we have assumed that all buyers and sellers have the same outside option of 0 and that buyers compete away their surplus. This is mainly done for the sake of exposition. Here, we describe what happens in the presence of endogenous entry.

Particularly, suppose that buyers' outside option is random and given by ν and is distributed according to a differentiable cumulative distribution function $G(\nu)$. We assume that sellers' outside option is 0. If we assume that the support of $G(\cdot)$ is the entire real line, then there must exist a threshold ν_e where buyers will buy the object if and only if their outside option is not greater than ν_e. Moreover, since our equilibrium concept is competitive equilibrium, there must exist a threshold θ_e where sellers produce if and only if $\theta \geq \theta_e$. In equilibrium the level of prices must adjust so that markets clear, i.e.,

$$G(\nu_e) = 1 - F(\theta_e)\,. \tag{20}$$

In essence, with random outside options for buyers, the overall level of prices is determined by the market clearing condition (20). The properties of optimal rating systems that

[23]See Jullien (2000) for treating participation constraints in classic mechanism design.

we have shown in section 4 hold independent of the division of the surplus between buyers and sellers. This implies that the properties of optimal rating systems that we discussed in section 4 go through even in the presence of endogenous entry. The only additional constraint that endogenous entry as modeled here imposes on optimal rating systems is that the rating system must punish the seller types below θ_e in such a way to discourage their entry.[24]

7 Conclusion

In this paper, we have studied the role and the design of rating systems in providing incentives for provision of quality. To solve the problem, we have showed a characterization results that establishes that sellers' second-order expectations are majorized by the sellers' true quality choices. This characterization result allows us provide fairly general characterization of a certain subset of Pareto optimal rating systems and draw general insights on optimal design of rating systems.

In our analysis, we have mainly focused on heterogeneity in quality in the form of vertical differentiation where all buyers value quality in the same fashion. Naturally, one can ask about the effect of horizontal differentiation. In such an environment, information provision improves the allocation and sorting of buyers among sellers with different quality. In a companion paper Saeedi and Shourideh (2020), we undertake the analysis of this problem. Additionally, one can argue that rating systems often use past performance to provide information to the market.[25] Designing rating systems in such a dynamic setting is an important problem which we leave for future work.

References

Albano, G. L. and A. Lizzeri (2001): "Strategic certification and provision of quality," *International economic review*, 42, 267–283.

Aliprantis, C. D. and K. Border (2013): *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer-Verlag Berlin and Heidelberg GmbH & Company KG.

Alonso, R. and O. Câmara (2016): "Persuading voters," *American Economic Review*, 106, 3590–3605.

Baron, D. P. and R. B. Myerson (1982): "Regulating a monopolist with unknown costs," *Econometrica: Journal of the Econometric Society*, 911–930.

Best, J. and D. Quigley (2020): "Persuasion for the long run," *Available at SSRN 2908115.*

[24]Mechanisms like this are employed in platforms such as Uber where drivers with low ratings are excluded.

[25]See Best and Quigley (2020) for some work along this line.

Blackwell, D. (1953): "Equivalent comparisons of experiments," *The annals of mathematical statistics*, 265–272.

Boleslavsky, R. and C. Cotton (2015): "Grading standards and education quality," *American Economic Journal: Microeconomics*, 7, 248–79.

Boleslavsky, R. and K. Kim (2020): "Bayesian persuasion and moral hazard," *Working Paper, Emory University.*

Doob, J. L. (1994): *Measure theory*, Springer Science & Business Media.

Doval, L. and V. Skreta (2019): "Mechanism design with limited commitment," *arXiv preprint arXiv:1811.03579.*

Dworczak, P., S. D. Kominers, and M. Akbarpour (forthcoming): "Redistribution Through Markets," *Econometrica.*

Dworczak, P. and G. Martini (2019): "The simple economics of optimal persuasion," *Journal of Political Economy*, 127, 1993–2048.

Gentzkow, M. and E. Kamenica (2016): "A Rothschild-Stiglitz Approach to Bayesian Persuasion," *American Economic Review*, 106.

Gershkov, A., B. Moldovanu, P. Strack, and M. Zhang (2020): "Optimal Auctions for Dual Risk Averse Bidders: Myerson meets Yaari," *Available at SSRN.*

Guesnerie, R. and J.-J. Laffont (1984): "A Complete Solution to a Class of Principal-Agent Probems with an Application to the Control of a Self-Managed Firm," *Journal of public Economics*, 25, 329–369.

Guo, Y. and E. Shmaya (2019): "The interval structure of optimal disclosure," *Econometrica*, 87, 653–675.

Harbaugh, R. and E. Rasmusen (2018): "Coarse grades: Informing the public by withholding information," *American Economic Journal: Microeconomics*, 10, 210–35.

Hardy, G., J. Littlewood, and G. Polya (1934): *Inequalities*, Cambridge Universtiy Press, Cambridge, UK.

Hardy, G., G. Polya, and J. Littlewood (1929): "Some Simple Inequalities Satisfied by Convex Functions," *Messenger of Mathematics*, 58, 145–152.

Hopenhayn, H. and M. Saeedi (2020): "Optimal Quality Ratings and Market Outcomes," *National Bureau of Economic Research Working Paper.*

Hui, X., M. Saeedi, G. Spagnolo, and S. Tadelis (2020): "Raising the Bar: Certification Thresholds and Market Outcomes,"," *Working Paper, Carnegie Mellon University.*

Jullien, B. (2000): "Participation constraints in adverse selection models," *Journal of Economic Theory*, 93, 1–47.

Kamenica, E. and M. Gentzkow (2011): "Bayesian persuasion," *American Economic Review*, 101, 2590–2615.

Kleiner, A., B. Moldovanu, and P. Strack (2020): "Extreme Points and Majorization: Economic Applications," *mimeo*.

Kolotilin, A. (2018): "Optimal information disclosure: A linear programming approach," *Theoretical Economics*, 13, 607–635.

Kolotilin, A., T. Mylovanov, A. Zapechelnyuk, and M. Li (2017): "Persuasion of a privately informed receiver," *Econometrica*, 85, 1949–1964.

Lizzeri, A. (1999): "Information revelation and certification intermediaries," *The RAND Journal of Economics*, 214–231.

Luenberger, D. G. (1997): *Optimization by Vector Space Methods*, John Wiley & Sons.

Marshall, A. W., I. Olkin, and B. C. Arnold (1979): *Inequalities: theory of majorization and its applications*, vol. 143, Springer.

Mussa, M. and S. Rosen (1978): "Monopoly and product quality," *Journal of Economic theory*, 18, 301–317.

Myerson, R. B. (1981): "Optimal auction design," *Mathematics of operations research*, 6, 58–73.

Ostrovsky, M. and M. Schwarz (2010): "Information disclosure and unraveling in matching markets," *American Economic Journal: Microeconomics*, 2, 34–63.

Rayo, L. and I. Segal (2010): "Optimal information disclosure," *Journal of political Economy*, 118, 949–987.

Roesler, A.-K. and B. Szentes (2017): "Buyer-optimal learning and monopoly pricing," *American Economic Review*, 107, 2072–80.

Rothschild, M. and J. E. Stiglitz (1970): "Increasing risk: I. A definition," *Journal of Economic theory*, 2, 225–243.

Royden, H. L. and P. Fitzpatrick (1988): *Real Analysis*, vol. 32, Macmillan New York.

Saeedi, M. and A. Shourideh (2020): "Rate to Match: Optimal Rating Design with Sorting," *Work in Progress*.

Shaked, M. and J. G. Shanthikumar (2007): *Stochastic orders*, Springer Science & Business Media.

Yaari, M. E. (1987): "The dual theory of choice under risk," *Econometrica*, 95–115.

Zapechelnyuk, A. (2020): "Optimal quality certification," *American Economic Review: Insights*, 2, 161–76.

Zubrickas, R. (2015): "Optimal grading," *International Economic Review*, 56, 751–776.

Appendix

A Proofs

A.1 Proof of Proposition 1

Proof. We first show that (5) holds:

Lemma 2. *Let* $\mathbf{q}$ *be a vector of qualities and* $\overline{\mathbf{q}}$ *be a vector of signaled qualities for an arbitrary rating system. Then, there exists an* $N \times N$ *positive matrix* $\mathbf{A}$ *such that*

$$\overline{\mathbf{q}} = \mathbf{A}\mathbf{q},$$

where the matrix $\mathbf{A}$ *satisfies*

$$\mathbf{f}^T\mathbf{A} = \mathbf{f}^T, \mathbf{A}\mathbf{e} = \mathbf{e}, \tag{21}$$

where $\mathbf{e} = (1, \cdots, 1)^T$.

Proof. Given (4), $\mathbf{A}$ is given by

$$\mathbf{A} = \text{diag}\,(\mathbf{f})^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau,$$

which is simply a rewriting of (4) in matrix form; diag $(\mathbf{f})$ is an $N \times N$ matrix that has $\mathbf{f}$ as diagonal and 0 otherwise. We have

$$\begin{aligned}\mathbf{A}\mathbf{e} &= \text{diag}\,(\mathbf{f})^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T \mathbf{e} d\tau \\ &= \text{diag}\,(\mathbf{f})^{-1} \int \boldsymbol{\mu} d\tau \\ &= \mathbf{e},\end{aligned}$$

where $\boldsymbol{\mu}^T\mathbf{e} = 1$ and $\mathbf{f} = \int \boldsymbol{\mu} d\tau$. Moreover,

$$\begin{aligned}\mathbf{f}^T\mathbf{A} &= \sum_i f_i \frac{1}{f_i} \int \mu_i \boldsymbol{\mu}^T d\tau \\ &= \sum_i \int \mu_i \boldsymbol{\mu}^T d\tau = \int \boldsymbol{\mu}^T d\tau = \mathbf{f}^T.\end{aligned}$$

□

Now suppose that matrix $\mathbf{A}$ is given by $\mathbf{A} = (a_{ij})_{i,j}$. We have that

$$\begin{aligned}
\sum_{i=1}^{k} f_i \overline{q}_i &= \sum_{i=1}^{k} f_i \sum_{j=1}^{N} a_{ij} q_j \\
&= \sum_{j=1}^{N} q_j \sum_{i=1}^{k} f_i a_{ij} \\
&\geq \sum_{j=1}^{k-1} q_j \sum_{i=1}^{k} f_i a_{ij} + q_k \sum_{j=k}^{N} \sum_{i=1}^{k} f_i a_{ij}. \qquad (22)
\end{aligned}$$

Since $\mathbf{A}$ satisfies (5), the following equality holds

$$\sum_{i=1}^{k} f_i \sum_{j=1}^{N} a_{ij} = \sum_{i=1}^{k} f_i.$$

Thus, we can write the above as

$$\begin{aligned}
\sum_{i=1}^{k} f_i \overline{q}_i &\geq \sum_{j=1}^{k-1} q_j \sum_{i=1}^{k} f_i a_{ij} + q_k \sum_{j=k}^{N} \sum_{i=1}^{k} f_i a_{ij} \\
&= \sum_{j=1}^{k-1} q_j \sum_{i=1}^{k} f_i a_{ij} + q_k \left[\sum_{i=1}^{k} f_i - \sum_{j=1}^{k-1} \sum_{i=1}^{k} f_i a_{ij} \right] \\
&= \sum_{j=1}^{k-1} q_j f_j + \sum_{j=1}^{k-1} q_j \left[\sum_{i=1}^{k} f_i a_{ij} - f_j \right] + \\
&\quad q_k \left[f_k + \sum_{j=1}^{k-1} f_j - \sum_{j=1}^{k-1} \sum_{i=1}^{k} f_i a_{ij} \right] \\
&= \sum_{j=1}^{k} q_j f_j + \sum_{j=1}^{k-1} (q_k - q_j) \left[f_j - \sum_{i=1}^{k} f_i a_{ij} \right] \\
&= \sum_{j=1}^{k} q_j f_j + \sum_{j=1}^{k-1} (q_k - q_j) \sum_{i=k+1}^{N} f_i a_{ij} \\
&\geq \sum_{j=1}^{k} q_j f_j,
\end{aligned}$$

where $q_k \geq q_j$ for all $j \leq k-1$ and $\mathbf{f}^T \mathbf{A} = \mathbf{f}^T$. Finally,

$$\mathbf{f}^T \overline{\mathbf{q}} = \mathbf{f}^T \mathbf{A} \mathbf{q} = \mathbf{f}^T \mathbf{q},$$

which concludes the proof. $\square$

A.2 Proof of Theorem 1

Proof. We define $\mathcal{S}$ as follows

$$\mathcal{S} = \left\{ \mathbf{r} | \exists \tau \in \Delta\left(\Delta\left(\Theta\right)\right), \mathbf{r} = \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \cdot \mathbf{q} \text{ with } \mathbf{f} = \int \boldsymbol{\mu} d\tau \right\}$$

This is obviously a compact set since $\mathcal{S} \subset [\min_i q_i, \max q_i]^N$. Moreover $\mathcal{S}$ is a convex set since if τ_1 and τ_2 satisfy Bayes plausibility, then so is their convex combination and $\text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau$ is linear in τ. We show that if $\overline{\mathbf{q}}$ satisfies the majorization property, then it must be a member of $\mathcal{S}$. To show this, we show that for any $\boldsymbol{\lambda} \in \mathbb{R}^N$, there exists $\mathbf{r} \in \mathcal{S}$ such that $\boldsymbol{\lambda}^T \overline{\mathbf{q}} \leq \boldsymbol{\lambda}^T \mathbf{r}$. Then if $\overline{\mathbf{q}} \notin \mathcal{S}$, then there must exist $\boldsymbol{\lambda} \in \mathbb{R}^N$ such that $\boldsymbol{\lambda}^T \overline{\mathbf{q}} > \boldsymbol{\lambda}^T \mathbf{r}, \forall \mathbf{r} \in \mathcal{S}$. This is in contradiction with the previous claim and so we must have that $\overline{\mathbf{q}} \in \mathcal{S}$.

Note that without loss of generality, we can assume that $\boldsymbol{\lambda} \geq 0$. This is because if $\boldsymbol{\lambda}^T \overline{\mathbf{q}} \leq \boldsymbol{\lambda}^T \mathbf{r}$ then for some $\alpha > 0$, we have

$$\begin{aligned}
\alpha \mathbf{f}^T \overline{\mathbf{q}} &= \alpha \mathbf{f}^T \mathbf{q} \\
\alpha \mathbf{f}^T \mathbf{r} &= \alpha \sum_i f_i \frac{1}{f_i} \int \boldsymbol{\mu}_i \boldsymbol{\mu}^T d\tau \cdot \mathbf{q} \\
&= \alpha \int \sum_i \boldsymbol{\mu}_i \boldsymbol{\mu}^T d\tau \cdot \mathbf{q} \\
&= \alpha \int \boldsymbol{\mu}^T d\tau \cdot \mathbf{q} \\
&= \alpha \mathbf{f}^T \mathbf{q}
\end{aligned}$$

and hence,

$$\left(\boldsymbol{\lambda} + \alpha \mathbf{f}\right)^T \overline{\mathbf{q}} \leq \left(\boldsymbol{\lambda} + \alpha \mathbf{f}\right)^T \mathbf{r}.$$

That is, a choice of α can guarantee that $\boldsymbol{\lambda} + \alpha \mathbf{f}$ has all elements positive.

We prove that for all $\boldsymbol{\lambda} \geq 0$ there exists $\mathbf{r} \in K$ such that $\boldsymbol{\lambda}^T \overline{\mathbf{q}} \leq \boldsymbol{\lambda}^T \mathbf{r}$ using induction on N.

When $N = 2$, there are two cases:

1. $\frac{\lambda_1}{\lambda_1 + \lambda_2} \geq f_1$. In this case,

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} \overline{q}_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} \overline{q}_2 \leq f_1 \overline{q}_1 + f_2 \overline{q}_2 = f_1 q_1 + f_2 q_2,$$

since $\overline{q}_1 \leq \overline{q}_2$. Thus, if we choose $\tau\left(\{\mathbf{f}\}\right) = 1$ – no information, then $\mathbf{r} = \left(\mathbf{f}^T \mathbf{q}, \mathbf{f}^T \mathbf{q}\right) \in \mathcal{S}$. The above inequality then implies that

$$\boldsymbol{\lambda}^T \overline{\mathbf{q}} \leq \left(\lambda_1 + \lambda_2\right) \mathbf{f}^T \mathbf{q} = \boldsymbol{\lambda}^T \mathbf{r}$$

which proves the claim.

2. $\frac{\lambda_1}{\lambda_1+\lambda_2} \le f_1$. Since $\overline{q}_1 \ge q_1$ and $\mathbf{f}^T\mathbf{q} = \mathbf{f}^T\overline{\mathbf{q}}$, we must have that $\overline{q}_2 \le q_2$. Therefore,

$$\overline{q}_2 - \overline{q}_1 \le q_2 - q_1$$

Multiplying both sides by $\left(\frac{\lambda_2}{\lambda_1+\lambda_2} - f_2\right) \ge 0$ we can write

$$\left(\frac{\lambda_2}{\lambda_1+\lambda_2} - f_2\right)(\overline{q}_2 - \overline{q}_1) \le \left(\frac{\lambda_2}{\lambda_1+\lambda_2} - f_2\right)(q_2 - q_1)$$

and we can add $f_1\overline{q}_1 + f_2\overline{q}_2 = f_1 q_1 + f_2 q_2$ to both sides of the above inequality and have

$$\begin{aligned}
\left(\frac{\lambda_2}{\lambda_1+\lambda_2} - f_2\right)(\overline{q}_2 - \overline{q}_1) + f_2\overline{q}_2 + f_1\overline{q}_1 &\le \left(\frac{\lambda_2}{\lambda_1+\lambda_2} - f_2\right)(q_2 - q_1) + f_2 q_2 + f_1 q_1 \\
\frac{\lambda_2}{\lambda_1+\lambda_2}\overline{q}_2 + \left[f_1 + f_2 - \frac{\lambda_2}{\lambda_1+\lambda_2}\right]\overline{q}_1 &\le \frac{\lambda_2}{\lambda_1+\lambda_2} q_2 + \left[f_1 + f_2 - \frac{\lambda_2}{\lambda_1+\lambda_2}\right] q_1 \\
\frac{\lambda_2}{\lambda_1+\lambda_2}\overline{q}_2 + \frac{\lambda_1}{\lambda_1+\lambda_2}\overline{q}_1 &\le \frac{\lambda_2}{\lambda_1+\lambda_2} q_2 + \frac{\lambda_1}{\lambda_1+\lambda_2} q_1
\end{aligned}$$

If we choose $\tau(\{\mathbf{e}_i\}) = f_i$, then τ satisfies Bayes plausibility and $\mathbf{p}' = \text{diag}(\mathbf{f})^{-1}\int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \mathbf{q} = \text{diag}(\mathbf{f})^{-1}\text{diag}(\mathbf{f})\mathbf{q} = \mathbf{q}$ and the above inequality implies $\boldsymbol{\lambda}^T\overline{\mathbf{q}} \le \boldsymbol{\lambda}^T\mathbf{p}'$ which proves the claim.

Now consider $\overline{\mathbf{q}}$, $\mathbf{q}$ and $\mathbf{f}$ and suppose that they satisfy the hypothesis of the claim. There are two possibilities:

Case 1. $\exists i \in \{1, \cdots, N\}$ such that $\lambda_i/f_i \le \lambda_{i-1}/f_{i-1}$. In this case, consider the following $N-1$ dimensional vectors

$$\begin{aligned}
\overline{\mathbf{q}}' &= \left(\overline{q}_1, \cdots, \overline{q}_{i-2}, \frac{f_{i-1}\overline{q}_{i-1} + f_i\overline{q}_i}{f_{i-1}+f_i}, \overline{q}_{i+1}, \cdots, \overline{q}_N\right) \\
\mathbf{q}' &= \left(q_1, \cdots, q_{i-2}, \frac{f_{i-1}q_{i-1} + f_i q_i}{f_{i-1}+f_i}, q_{i+1}, \cdots, q_N\right) \\
\mathbf{f}' &= (f_1, \cdots, f_{i-2}, f_{i-1}+f_i, f_{i+1}, \cdots, f_N) \\
\boldsymbol{\lambda}' &= (\lambda_1, \cdots, \lambda_{i-2}, \lambda_{i-1}+\lambda_i, \lambda_{i+1}, \cdots, \lambda_N)
\end{aligned}$$

We have that

$$\sum_{j=1}^{k} \tilde{f}_j\overline{q}'_j = \begin{cases} \sum_{j=1}^{k} f_j\overline{q}_j & k \le i-2 \\ \sum_{j=1}^{k+1} f_j\overline{q}_j & k \ge i-1 \end{cases}$$

and a similar property holds for $\tilde{\mathbf{q}}$. This implies that $\overline{\mathbf{q}}', \mathbf{q}'$ and $\mathbf{f}'$ satisfy the hypothesis of our claim and as a result and by the induction hypothesis there exists $\tau' \in \Delta(\Delta(\Theta'))$ – with $\Theta' = \left\{1, 2, \cdots, i-2, \hat{i}, i+1, \cdots, N\right\}$ – so that

$$\mathbf{f}' = \int \boldsymbol{\mu} d\tau', \mathbf{r}' = \text{diag}(\mathbf{f}')^{-1}\int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau'\mathbf{q}', (\boldsymbol{\lambda}')^T\overline{\mathbf{q}}' \le (\boldsymbol{\lambda}')^T\mathbf{r}' \tag{23}$$

We construct $\tau \in \Delta(\Delta(\Theta))$ from τ' by assuming that τ always sends the same signal for q_{i-1} and q_i as τ'. Formally, we define a subset K of $\Delta(\Theta)$ as follows:

$$K = \left\{ \boldsymbol{\mu} \in \Delta(\Theta) \,|\, \exists \mu' \in \Delta(\Theta'), \mu_j = \mu'_j, j \neq i, i-1, \mu_{i-1} = \frac{f_{i-1}}{f_{i-1}+f_i}\mu'_{\hat{i}}, \mu_i = \frac{f_i}{f_{i-1}+f_i}\mu'_{\hat{i}} \right\}$$

This is a borel subset of $\Delta(\Theta)$ – the set of beliefs where $\mu_i/\mu_{i-1} = f_i/f_{i-1}$. Moreover, for any borel subset A of K, we can define its projection $P(A)$ in $\Delta(\Theta')$ as $P(A) = \left\{ \mu' | \exists \mu \in A, \mu'_j = \mu_j, j \neq \hat{i}, \mu'_{\hat{i}} = \mu_{i-1} + \mu_i \right\}$. Given this, we define

$$\tau(A) = \tau'(P(A \cap K)).$$

In words, the above information structure keeps the receiver fully uninformed about states i and $i-1$ since their relative probabilities are always equal to the relative probability of the prior. We have

$$\begin{aligned}
\int \boldsymbol{\mu} d\tau = \int_K \boldsymbol{\mu} d\tau &= \int_{\Delta(\Theta')} \left(\mu_1, \cdots, \mu_{i-2}, \frac{f_{i-1}}{f_{i-1}+f_i}\mu_{\hat{i}}, \frac{f_i}{f_{i-1}+f_i}\mu_{\hat{i}}, \mu_{i+1}, \cdots, \mu_N \right)^T d\tau' \\
&= \left(\int \mu_1 d\tau', \cdots, \int \mu_{i-2} d\tau', \frac{f_{i-1}}{f_{i-1}+f_i} \int \mu_{\hat{i}} d\tau', \frac{f_i}{f_{i-1}+f_i} \int \mu_{\hat{i}} d\tau', \int \mu_{i+1} d\tau', \cdots, \int \mu_N d\tau' \right)^T \\
&= \left(f_1, \cdots, f_{i-2}, \frac{f_{i-1}}{f_{i-1}+f_i}(f_{i-1}+f_i), \frac{f_i}{f_{i-1}+f_i}(f_{i-1}+f_i), f_{i+1}, \cdots, f_N \right)^T = \mathbf{f}
\end{aligned}$$

where in the above we have used the fact that τ' satisfies (23).
Moreover, we have

$$\text{diag}(\mathbf{f})^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau = \left(\frac{1}{f_k} \int \mu_k \mu_j d\tau \right)_{k,j \in \{1,\cdots,N\}} = \begin{cases} \frac{1}{f_k} \int \mu_k \mu_j d\tau' & k, j \neq i, i-1 \\ \frac{1}{f_i + f_{i-1}} \int \mu_{\hat{i}} \mu_j d\tau' & k \in \{i, i-1\}, j \neq i, i-1 \\ \frac{f_j}{f_k(f_i+f_{i-1})} \int \mu_k \mu_{\hat{i}} d\tau' & k \neq i, i-1, j \in \{i, i-1\} \\ \frac{f_j}{(f_i+f_{i-1})^2} \int (\mu_{\hat{i}})^2 d\tau' & k, j \in \{i, i-1\} \end{cases}$$

Therefore, if we let $\mathbf{r} = \text{diag}(\mathbf{f})^{-1} \left(\int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \right) \mathbf{q}$, we have

$$r_k = \frac{1}{f_k} \sum_{j=1}^{N} \int \mu_k \mu_j d\tau q_j$$

When $k \neq i, i-1$, we can write the above as

$$\begin{aligned}
r_k &= \frac{1}{f_k} \sum_{j \neq i, i-1} \int \mu_k \mu_j d\tau' q_j \\
&+ \frac{1}{f_k} \frac{f_{i-1}}{f_i + f_{i-1}} \int \mu_k \mu_{\hat{i}} d\tau' q_{i-1} + \frac{1}{f_k} \frac{f_i}{f_i + f_{i-1}} \int \mu_k \mu_{\hat{i}} d\tau' q_i \\
&= \frac{1}{f_k} \sum_{j \neq i, i-1} \int \mu_k \mu_j d\tau' q_j + \frac{1}{f_k} \int \mu_k \mu_{\hat{i}} d\tau' q_{i-1} \frac{f_i q_i + f_{i-1} q_{i-1}}{f_i + f_{i-1}} \\
&= \frac{1}{f_k} \sum_{j \in \hat{\Theta}} \int \mu_k \mu_j d\tau' \tilde{q}_j = r'_k
\end{aligned}$$

where in the above we have used the fact that $q_j = q'_j$ for all $j \neq i, i-1$ and $q'_{\hat{i}} = \frac{f_i q_i + f_{i-1} q_{i-1}}{f_{i-1} + f_i}$ and the definition of $\mathbf{r}$. If $k = i, i-1$, then

$$\begin{aligned}
r_k &= \frac{1}{f_i + f_{i-1}} \sum_{j \neq i, i-1} \int \mu_{\hat{i}} \mu_j d\tau' q_j \\
&+ \frac{f_i}{(f_i + f_{i-1})^2} \int (\mu_{\hat{i}})^2 d\tau' q_i + \frac{f_{i-1}}{(f_i + f_{i-1})^2} \int (\mu_{\hat{i}})^2 d\tau' q_{i-1} \\
&= \frac{1}{f_i + f_{i-1}} \sum_{j \neq i, i-1} \int \mu_{\hat{i}} \mu_j d\tau' q_j \\
&+ \frac{1}{(f_i + f_{i-1})} \int (\mu_{\hat{i}})^2 d\tau' \frac{f_i q_i + f_{i-1} q_{i-1}}{f_i + f_{i-1}} \\
&= \frac{1}{f'_{\hat{i}}} \sum_{j \in \hat{\Theta}} \int \mu_{\hat{i}} \mu_j d\tau' \tilde{q}_j = r'_{\hat{i}}
\end{aligned}$$

This implies that

$$\begin{aligned}
\boldsymbol{\lambda}^T \mathbf{r} &= \sum_{j=1}^{N} \lambda_j r_j \\
&= \sum_{j \neq i, i-1} \lambda_j r'_j + (\lambda_i + \lambda_{i-1}) r'_{\hat{i}} \\
&= \sum_{j = i, i-1} \lambda'_j r'_j + \lambda'_{\hat{i}} r'_{\hat{i}} \\
&= (\boldsymbol{\lambda}')^T \mathbf{r}'
\end{aligned}$$

Moreover,

$$\begin{aligned}
\boldsymbol{\lambda}^T \overline{\mathbf{q}} &= \sum_{j=1}^{N} \lambda_j \overline{q}_j \\
&= \sum_{j \neq i, i-1} \lambda_j \overline{q}_j + \lambda_{i-1} \overline{q}_{i-1} + \lambda_i \overline{q}_i
\end{aligned}$$

We can write

$$\lambda_{i-1}\overline{q}_{i-1} + \lambda_i \overline{q}_i = f_{i-1}\frac{\lambda_{i-1}}{f_{i-1}}\overline{q}_{i-1} + f_i \frac{\lambda_i}{f_i}\overline{q}_i$$

Since

$$\frac{\lambda_{i-1}}{f_{i-1}} \geq \frac{\lambda_i}{f_i}, \overline{q}_{i-1} \leq \overline{q}_i$$

Chebyshev's sum inequality implies that

$$\begin{aligned}\frac{f_{i-1}}{f_{i-1}+f_i}\frac{\lambda_{i-1}}{f_{i-1}}\overline{q}_{i-1} + \frac{f_i}{f_i+f_{i-1}}\frac{\lambda_i}{f_i}\overline{q}_i &\leq \left(\frac{f_{i-1}}{f_{i-1}+f_i}\frac{\lambda_{i-1}}{f_{i-1}} + \frac{f_i}{f_i+f_{i-1}}\frac{\lambda_i}{f_i}\right) \times \\ &\quad \left(\frac{f_{i-1}}{f_{i-1}+f_i}\overline{q}_{i-1} + \frac{f_i}{f_i+f_{i-1}}\overline{q}_i\right) \\ &= \frac{\lambda_{i-1}+\lambda_i}{f_{i-1}+f_i}\frac{f_{i-1}\overline{q}_{i-1}+f_i\overline{q}_i}{f_{i-1}+f_i} \\ &= \frac{\lambda'_{\hat{i}}}{f_{i-1}+f_i}\overline{q}'_{\hat{i}}\end{aligned}$$

That is

$$\lambda_{i-1}\overline{q}_{i-1} + \lambda_i\overline{q}_i \leq \lambda'_{\hat{i}}\overline{q}'_{\hat{i}}$$

We can therefore write

$$\boldsymbol{\lambda}^T\overline{\mathbf{q}} \leq (\boldsymbol{\lambda}')^T\,\overline{\mathbf{q}}'$$

and as a result

$$\boldsymbol{\lambda}^T\overline{\mathbf{q}} \leq (\boldsymbol{\lambda}')^T\,\overline{\mathbf{q}}' \leq (\boldsymbol{\lambda}')^T\,\mathbf{r}' = \boldsymbol{\lambda}^T\mathbf{r}$$

which establishes the claim.

Case 2. $\forall i \in \{1, \cdots, N\}, \frac{\lambda_{i-1}}{f_{i-1}} \leq \frac{\lambda_i}{f_i}$. Then we can write

$$\begin{aligned}\boldsymbol{\lambda}^T\overline{\mathbf{q}} &= \sum_{i=1}^{N}\lambda_i p_i = \sum_{i=1}^{N}\frac{\lambda_i}{f_i}f_i p_i \\ &= \sum_{i=1}^{N}\left(\frac{\lambda_i}{f_i} - \frac{\lambda_{i-1}}{f_{i-1}}\right)\sum_{j=i}^{N} f_j p_j \\ &= \sum_{i=1}^{N}\left(\frac{\lambda_i}{f_i} - \frac{\lambda_{i-1}}{f_{i-1}}\right)\left[\mathbf{f}^T\overline{\mathbf{q}} - \sum_{j=1}^{i-1} f_j p_j\right] \\ &\leq \sum_{i=1}^{N}\left(\frac{\lambda_i}{f_i} - \frac{\lambda_{i-1}}{f_{i-1}}\right)\left[\mathbf{f}^T\mathbf{q} - \sum_{j=1}^{i-1} f_j q_j\right] \\ &= \sum_{i=1}^{N}\left(\frac{\lambda_i}{f_i} - \frac{\lambda_{i-1}}{f_{i-1}}\right)\sum_{j=i}^{N} f_j q_j \\ &= \sum_{i=1}^{N}\frac{\lambda_i}{f_i}f_i q_i = \boldsymbol{\lambda}^T\mathbf{q}\end{aligned}$$

Note that $\mathbf{q}$ is the vector of expected signaled qualities under full information, i.e., $\tau(\{\mathbf{e}_i\}) = f_i$ and thus $\mathbf{q} \in \mathcal{S}$. This completes the proof. □

A.3 Proof of Proposition 2

Proof. The proof of incentive compatibility and the "only if" direction of the claim is straightforward and is omitted. Here, we prove the "if" part. In other words, consider a pair of functions $\{q(\theta), \overline{q}(\theta)\}_{\theta\in\Theta}$ that satisfy conditions 1 and 2 in the statement of proposition.

Consider a sequence of partitions of Θ given by $\Theta^n = \left\{\underline{\theta} = \theta_0^n < \theta_1^n < \cdots < \theta_n^n = \overline{\theta}\right\}$ for $n = 1, 2, \cdots$ with $\min_{i:0\leq i\leq n-1} \theta_{i+1}^n - \theta_i^n \to 0$ and $\Theta^n \subset \Theta^{n+1}$. Define $\mathbf{f}^n, \mathbf{q}^n, \overline{\mathbf{q}}^n$ as follows

$$
\begin{aligned}
f_i^n &= F(\theta_i^n -) - F\left(\theta_{i-1}^n\right), 1 \leq i \leq n-1 \\
f_n^n &= F\left(\overline{\theta}\right) - F\left(\theta_{n-1}^n\right) \\
q_i^n &= \begin{cases} \frac{\int_{\theta_{i-1}^n}^{\theta_i^n} q(\theta) dF(\theta)}{f_i^n} & \text{if } f_i^n > 0 \\ q\left(\theta_{i-1}^n\right) & \text{if } f_i^n = 0 \end{cases} \\
\overline{q}_i^n &= \begin{cases} \frac{\int_{\theta_{i-1}^n}^{\theta_i^n} \overline{q}(\theta) dF(\theta)}{f_i^n} & \text{if } f_i^n > 0 \\ \overline{q}\left(\theta_{i-1}^n\right) & \text{if } f_i^n = 0 \end{cases}
\end{aligned}
$$

Let $\overline{q}^n$ and q^n represent the discrete random variables whose values are given by $\overline{q}_i^n$ and q_i^n with probability f_i^n. Given that $q(\theta)$ and $\overline{q}(\theta)$ are increasing functions, then q_i^n and $\overline{q}_i^n$ are increasing in i. Moreover, by construction, $\overline{\mathbf{q}}^n \succcurlyeq_{\mathbf{F}^n} \mathbf{q}^n$. Thus, we can use Theorem 1 and an information structure (π^n, S^n) exists where $\pi^n : \Theta^n \to \Delta(S^n)$ exists under which $\overline{q}_i^n = \mathbb{E}\left[\mathbb{E}\left[q_i^n|s\right]|q_i^n\right]$. Note that each (π^n, S^n) induces a distribution over posterior beliefs of the buyers given by $\tau^n \in \Delta(\Delta(\Theta^n))$. Note that since any measure in $\Delta(\Theta^n)$ can be embedded in $\Delta(\Theta)$. This is because for any $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n) \in \Delta(\Theta^n)$ we can construct $\hat{\mu} \in \Delta(\Theta)$ defined by $\hat{\mu}(A) = \sum_{i=1}^n \mu_i \mathbf{1}\left[\theta_i^n \in A\right]$ where A is an arbitrary Borel subset of Θ. Similarly, we can find $\hat{\tau}^n \in \Delta(\Delta(\Theta))$ which is equivalent to τ^n.

Now consider the random variable representing the joint distribution of θ^n and posterior mean $\mu[q] = \int q(\theta) d\mu$ for any $\mu \in \text{Supp}(\hat{\tau}^n)$. Let this be given by $\zeta^n = (q^n, \mu[q])$ where $\zeta^n \in \Delta(q(\Theta) \times q(\Theta))$ – where $q(\Theta) = \{q(\theta) | \theta \in \Theta\}$. By an application of Reisz Representation theorem (see Theorem 14.12 in Aliprantis and Border (2013)), $\Delta(q(\Theta) \times q(\Theta))$ is compact according to the weak-* topology.[26] This implies that the sequence $\{\zeta^n\}$ must

[26] A rough argument for sequential compactness of $\Delta(q(\Theta) \times q(\Theta))$ is as follows: Note that $C(X)$, the space of all continuous functions on $X = q(\Theta) \times q(\Theta)$, is separable since X is a compact, metrizable, and Hausdorff space (see Reisz's Theorem in Royden and Fitzpatrick (1988) – section12.3, page 251.) This implies that there exists a countable subset $\{f_i\}_{i=1}^\infty$ of $C(X)$ which is dense in $C(X)$ according to sup-norm. Thus, for any sequence of measures $\{\mu_m\}_{m=1}^\infty$ in $\Delta(X)$, for a given i, the sequence $\{\mu_m(f_i)\}_{m=1}^\infty$ where $\mu(f) = \int f(\theta) d\mu$ must have a convergent subsequent. Iterating repeatedly, as we increase i, we can find a subsequence $\{\mu_{m_k}\}_{k=1}^\infty$ where $\{\mu_{m_k}(f_i)\}$ converges. We define $\zeta(f_i) = \lim_{k\to\infty} \mu_{m_k}(f_i)$. Since $\{f_i\}$ is dense in $C(X)$, then $\zeta(f) = \lim_{k\to\infty} \mu_{m_k}(f)$ must exists for all $f \in C(X)$ and can be similarly defined.

have a convergent subsequence whose limit is given by $\zeta \in \Delta(q(\Theta) \times q(\Theta))$. Let $\mathcal{G}^n$ be the σ-field generated by the sets $\left\{\left[q\left(\theta_i^n\right), q\left(\theta_{i+1}^n\right)\right)\right\}_{i \leq n-1} \cup \left\{\left[q\left(\theta_{n-1}^n\right), q\left(\overline{\theta}\right)\right]\right\}$ and let $\mathcal{F}^n = \mathcal{G}^n \times \{\emptyset, \Delta(q(\Theta))\}$. In words, $\mathcal{F}^n$ conveys the information that $q(\theta) \in \left[q\left(\theta_i^n\right), q\left(\theta_{i+1}^n\right)\right)$ or $q(\theta) \in \left[q\left(\theta_{n-1}^n\right), q\left(\overline{\theta}\right)\right]$. Note that $\mathcal{F}^n \subset \mathcal{F}^{n+1}$ because $\Theta^n \subset \Theta^{n+1}$. Moreover,

$$\mathbb{E}\left[\zeta^n | \mathcal{F}^n\right] = \left(q^n, \overline{q}^n\right)$$

where the above holds by the construction of τ^n and ζ^n. As a result

$$\begin{aligned}
\mathbb{E}\left[\zeta^{n+1} | \mathcal{F}^n\right] &= \mathbb{E}\left[\mathbb{E}\left[\zeta^{n+1} | \mathcal{F}^{n+1}\right] | \mathcal{F}^n\right] \\
&= \mathbb{E}\left[\left(q^{n+1}, \overline{q}^{n+1}\right) | \mathcal{F}^n\right] \\
&= \left(q^n, \overline{q}^n\right)
\end{aligned}$$

where the last equality follows because $\mathbb{E}\left[\overline{q}(\theta) | \mathcal{F}^n\right] = \overline{q}^n, \mathbb{E}\left[q(\theta) | \mathcal{F}^n\right] = q^n$ given the definition of $\overline{q}^n$ and q^n above. All of this implies that $\mathcal{F}^n$ is a filtration and $(\zeta^n, \mathcal{F}^n)$ forms a bounded martingale – for a definition see Doob (1994). Hence by Doob's martingale convergence theorem – see Theorem XI.14 in Doob (1994), we must have that

$$\lim_{n \to \infty} \mathbb{E}\left[\zeta^n | \mathcal{F}^n\right] = \mathbb{E}\left[\zeta | \mathcal{F}\right]$$

Therefore, $\mathbb{E}_\zeta\left[\mu[q] | q(\theta)\right] = \overline{q}(\theta)$. This concludes the proof. ☐

A.4 Proof of Proposition 3

Proof. We prove the claim for a discrete distribution. The general claim follows from arguments similar to those made in the proof of Proposition 2.

Suppose that majorization inequality holds for some $k < N$, i.e., we have

$$\sum_{i=1}^k f_i \overline{q}_i = \sum_{i=1}^k f_i q_i$$

Recall the proof of Proposition 1. If for all $j > k$, $q_j > q_k$, then the above equality implies that the inequality (22) must hold with equality. As a result, we must have that $a_{ij} = 0$ for all $j \geq k+1, i \leq k$. Note that by definition of a_{ij} , it is given by

$$a_{ij} = \sum_{s \in S} \frac{\pi(\{s\} | q_i) \pi(\{s\} | q_j) f_j}{\sum_{l=1}^N \pi(\{s\} | q_l) f_l}$$

where in the above $S = \cup_{l=1}^N \text{Supp}\left(\pi(\cdot | q_l)\right)$. Hence, for all $i \leq k, j \geq k+1, \pi(\{s\} | q_i) \pi(\{s\} | q_j) = 0$. This implies that $\cup_{i \leq k} \text{Supp}\left(\pi(\cdot | q_i)\right) \cap \cup_{j \geq k+1} \text{Supp}\left(\pi(\cdot | q_j)\right) = \emptyset$ which establishes the claim. ☐

It is easy to show that $\zeta(f)$ is a linear functional over $C(X)$ and thus a member of its dual, $C(X)^*$. Hence, there must exists a measure $\hat{\zeta} \in \Delta(X)$ where $\zeta(f) = \int f d\hat{\zeta}$. This implies that μ_{m_k} converges to $\hat{\zeta}$ according to the weak-* topology and hence, $\Delta(X)$ is sequentially compact.

A.5 Proof of Proposition 4

Proof. Consider the relaxed optimization problem given by

$$\max \int_{\underline{\theta}}^{\overline{\theta}} \lambda(\theta)\,\Pi(\theta)\,dF(\theta)$$

subject to

$$\Pi'(\theta) = -C_\theta(q(\theta),\theta), \forall\theta$$
$$q(\theta): \text{ increasing}$$
$$\int_{\underline{\theta}}^{\overline{\theta}} \Pi(\theta)\,dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} [q(\theta) - C(q(\theta),\theta)]\,dF(\theta)$$
$$\Pi(\theta) \geq 0$$

We show that the solution to the above optimization satisfies the majorization constraint. Given incentive compatibility, we can calculate $\Pi(\underline{\theta})$ using integration by parts

$$\Pi(\underline{\theta}) = \int_{\underline{\theta}}^{\overline{\theta}} \left[q(\theta) - C(q(\theta),\theta) + C_\theta(q(\theta),\theta)\frac{1-F(\theta)}{f(\theta)}\right] dF(\theta)$$

Hence, the objective becomes

$$\int_{\underline{\theta}}^{\overline{\theta}} \left\{ q(\theta) - C(q(\theta),\theta) + C_\theta(q(\theta),\theta)\frac{[1-F(\theta)] - \int_\theta^{\overline{\theta}} \lambda(\theta')\,dF(\theta')}{f(\theta)} \right\} dF(\theta) \tag{24}$$

Note that since $\int \lambda(\theta)\,dF = 1$ and $\lambda(\theta)$ is decreasing, we have

$$1 > \frac{\int_\theta^{\overline{\theta}} \lambda(\theta')\,dF(\theta')}{1-F(\theta)}, \forall\theta$$

Now suppose that for an interval $I = [\theta_1, \theta_2]$ of θ's, $C_q(q(\theta),\theta) > 1$ at the optimum. If over this interval, $q(\theta)$ is strictly increasing, then we can reduce $q(\theta)$ such that at its lower end, $q(\theta)$ does not decrease. If $C_q > 1$, a perturbation of $q(\theta)$ given by $\delta q(\theta) < 0$ changes the objective by

$$\int_I \left\{ 1 - C_q(q(\theta),\theta) + C_{\theta q}(q(\theta),\theta)\frac{1-F(\theta) - \int_\theta^{\overline{\theta}} \lambda(\theta')\,dF(\theta')}{f(\theta)} \right\} \delta q(\theta)\,dF(\theta)$$

We have

$$C_{\theta q}(q(\theta),\theta) < 0, \delta q(\theta) < 0, \frac{1-F(\theta) - \int_\theta^{\overline{\theta}} \lambda(\theta')\,dF(\theta')}{f(\theta)} > 0$$

Therefore,

$$\int_I C_{\theta q}\left(q\left(\theta\right),\theta\right)\frac{1-F\left(\theta\right)-\int_{\theta}^{\overline{\theta}}\lambda\left(\theta'\right)dF\left(\theta'\right)}{f\left(\theta\right)}\delta q\left(\theta\right)dF>0$$

Moreover,

$$\int\left[1-C_q\left(q\left(\theta\right),\theta\right)\right]\delta q\left(\theta\right)dF>0$$

Thus, this perturbation increases the objective. Therefore, we cannot have $C_q > 1$ at the optimum. Thus, we have $C_q \leq 1$. If on the other hand, $q\left(\theta\right)$ is constant over an interval of the form $\left[\theta_1,\hat{\theta}\right]$, we can find the lowest θ_0 for which $q\left(\theta_0\right)=q\left(\theta_1\right)$. Since $C_q > 1$ over I, either there is a discontinuity at θ_0 in which case the above argument works or $C_q > 1$ even for values below θ_0. In this case, we extend I below θ_0 and repeat the above perturbation.

From the incentive constraint, we have

$$\overline{q}'\left(\theta\right)=C_q\left(q\left(\theta\right),\theta\right)q'\left(\theta\right)\leq q'\left(\theta\right)$$

Therefore, the function $\overline{q}\left(\theta\right)-q\left(\theta\right)$ is a weakly decreasing function. This implies that

$$\frac{\int_{\underline{\theta}}^{\theta}\left[\overline{q}\left(\theta'\right)-q\left(\theta'\right)\right]dF\left(\theta'\right)}{F\left(\theta\right)}\geq\frac{\int_{\theta}^{\overline{\theta}}\left[\overline{q}\left(\theta'\right)-q\left(\theta'\right)\right]dF\left(\theta'\right)}{1-F\left(\theta\right)}$$

Hence,

$$\begin{aligned}
0&=\int_{\underline{\theta}}^{\theta}\left[\overline{q}\left(\theta'\right)-q\left(\theta'\right)\right]dF\left(\theta'\right)+\int_{\theta}^{\overline{\theta}}\left[\overline{q}\left(\theta'\right)-q\left(\theta'\right)\right]dF\left(\theta'\right)\\
&\leq\int_{\underline{\theta}}^{\theta}\left[\overline{q}\left(\theta'\right)-q\left(\theta'\right)\right]dF\left(\theta'\right)+\left(1-F\left(\theta\right)\right)\frac{\int_{\underline{\theta}}^{\theta}\left[\overline{q}\left(\theta'\right)-q\left(\theta'\right)\right]dF\left(\theta'\right)}{F\left(\theta\right)}\\
&=\frac{\int_{\underline{\theta}}^{\theta}\left[\overline{q}\left(\theta'\right)-q\left(\theta'\right)\right]dF\left(\theta'\right)}{F\left(\theta\right)}\left(1+\frac{1-F\left(\theta\right)}{F\left(\theta\right)}\right)\\
&=\int_{\underline{\theta}}^{\theta}\left[\overline{q}\left(\theta'\right)-q\left(\theta'\right)\right]dF\left(\theta'\right)\left(1+\frac{1-F\left(\theta\right)}{F\left(\theta\right)}\right)\\
&=\int_{\underline{\theta}}^{\theta}\left[\overline{q}\left(\theta'\right)-q\left(\theta'\right)\right]dF\left(\theta'\right)\frac{1}{F\left(\theta\right)}
\end{aligned}$$

which implies that the allocation satisfies the majorization constraint.

The above proof also illustrates that when $C\left(\cdot,\cdot\right)$ is strictly sub-modular, then it must be that $C_q < 1$ for all values of θ. This is because the integrand in objective in (24) is strictly decreasing in $q\left(\theta\right)$ at $q^{FB}\left(\theta\right)$. This concludes the proof. □

A.6 Proof of Proposition 5

Proof. We show that first best allocation is the solution to relaxed problem where the incentive constraint is replaced with

$$\Pi(\theta) - \Pi(\underline{\theta}) \leq -\int_{\underline{\theta}}^{\theta} C_\theta(q(\theta'), \theta')\, d\theta' \tag{25}$$

Suppose to the contrary that at the optimum, there is an interval of θ's such that the majorization constraint is slack. Let's consider such an interval $I = (\theta_1, \theta_2)$ and assume that the majorization constraint binds at θ_1 and θ_2. Such interval must exists since $\int_{\underline{\theta}}^{\theta} [\overline{q}(\theta') - q(\theta')]\, dF(\theta')$ is continuous function of θ and as a result the set of θ's for which it takes positive values is an open set. Hence, it must be a countable union of disjoint intervals.

We show a contradiction in a few steps:

Step 1. For any subinterval of I, (25) cannot be slack. Suppose, to the contrary, that this is the case and that there is a subinterval $I' \subset I$ in which (25) is slack. Let $\hat{\theta}$ be the mid-point of I'. Then consider the following perturbation

$$\delta\overline{q}(\theta) = \begin{cases} -\varepsilon' & \theta \in I', \theta < \hat{\theta} \\ \varepsilon & \theta \in I', \theta \geq \hat{\theta} \\ 0 & \theta \notin I' \end{cases}$$

where $-\varepsilon'\left[F\left(\hat{\theta}\right) - F(\min I')\right] + \varepsilon\left[F(\max I') - F\left(\hat{\theta}\right)\right] = 0$ and $\varepsilon, \varepsilon' > 0$. Since majorization is slack over I', there exists a value of ε and ε' small enough so that this perturbation does not violate the majorization constraint. Moreover, (25) is slack over I' there exists a value of ε and ε' small enough so that (25) is satisfied. As a result, the perturbed allocation is still feasible and satisfies the constraints. The change in the objective resulting from this perturbation is given by

$$\begin{aligned}
\int_{I'} \lambda(\theta)\, \delta\Pi(\theta)\, dF(\theta) &= -\varepsilon' \int_{\min I'}^{\hat{\theta}} \lambda(\theta)\, dF(\theta) + \varepsilon \int_{\hat{\theta}}^{\max I'} \lambda(\theta)\, dF(\theta) \\
&= \varepsilon \left[\int_{\hat{\theta}}^{\max I'} \lambda(\theta)\, dF(\theta) - \frac{F(\max I') - F\left(\hat{\theta}\right)}{F\left(\hat{\theta}\right) - F(\min I')} \int_{\min I'}^{\hat{\theta}} \lambda(\theta)\, dF(\theta)\right] \\
&= \varepsilon \left(F(\max I') - F\left(\hat{\theta}\right)\right) \left[\frac{\int_{\hat{\theta}}^{\max I'} \lambda(\theta)\, dF(\theta)}{F(\max I') - F\left(\hat{\theta}\right)} - \frac{\int_{\min I'}^{\hat{\theta}} \lambda(\theta)\, dF(\theta)}{F\left(\hat{\theta}\right) - F(\min I')}\right] > 0
\end{aligned}$$

where the last inequality holds because $\lambda(\theta)$ is strictly increasing. The above implies the required contradiction since this perturbation increases the objective.

Step 2. If $q(\theta)$ is strictly increasing over a subinterval of I, then it must be that $C_q(q(\theta), \theta) \geq 1$. Suppose not. Then, since $q(\theta)$ is strictly increasing over

$I' \subset I$, it is possible to find a perturbation $\delta q(\theta)$ of $q(\theta)$ with $\delta q(\max I') = 0$ and $\delta q(\theta) > 0, \forall \theta \in I'/\{\max I'\}$ which keeps $q(\theta)$ monotone – see Figure 3. Let $\delta \overline{q}(\theta)$ be given by

$$\delta \overline{q}(\theta) = \begin{cases} \delta q(\theta) C_q(q(\theta), \theta) + \int_{I'} [1 - C_q(q(\theta), \theta)] \delta q(\theta) dF(\theta) & \theta \in I' \\ \int_{I'} [1 - C_q(q(\theta), \theta)] \delta q(\theta) dF(\theta) & \theta \notin I' \end{cases}$$

We have

$$\forall \theta \in I', \delta\Pi(\theta) = \delta\overline{q}(\theta) - C_q(q(\theta), \theta)\delta q(\theta) = \int_{I'} [1 - C_q(q(\theta), \theta)] \delta q(\theta) dF(\theta)$$

$$\forall \theta \notin I', \delta\Pi(\theta) = \int_{I'} [1 - C_q(q(\theta), \theta)] \delta q(\theta) dF(\theta)$$

This implies that the perturbation keeps the LHS of (25) unchanged. Since $C_{\theta q} \leq 0$, the perturbation increases the RHS of (25) and as a result the inequality (25) is satisfied for this perturbed allocation. Moreover, since majorization is slack over I' for a small enough perturbation $\delta q(\theta)$ it is still satisfied. This perturbation increases the profits of all sellers while it keep buyers' utility unchanged. This implies the required contradiction.

Step 3. We show that the above two statements lead to a contradiction. Since (25) binds for all values of θ except for a measure zero set, it must be that $\Pi(\theta)$ is almost everywhere monotone and as a result almost everywhere differentiable. We thus have

$$\Pi'(\theta) = -C_\theta(q(\theta), \theta)$$

which then implies

$$\overline{q}'(\theta) = C_q(q(\theta), \theta) q'(\theta) \geq q'(\theta)$$

Since majorization binds at θ_1 and θ_2, we must have that $\overline{q}(\theta_1) \geq q(\theta_1)$ and $\int_{\theta_1}^{\theta_2} [\overline{q}(\theta) - q(\theta)] dF(\theta) = 0$. Therefore, we must have that $\overline{q}(\theta) \geq q(\theta)$ for almost all values of $\theta \in I$. Since $\int_{\theta_1}^{\theta_2} [\overline{q}(\theta) - q(\theta)] dF(\theta) = 0$, we must have that $\overline{q}(\theta) = q(\theta)$ for almost all values of $\theta \in I$. This in turn implies that majorization is binding for almost all values of $\theta \in I$ which is a contradiction.

The above arguments establishes that the majorization constraint must be binding for all values of θ. Hence, $q(\theta) = \overline{q}(\theta)$ for all values of θ and thus the objective is maximized at $q(\theta) = q^{FB}(\theta)$ which concludes the proof. □

A.7 Proof of Proposition 6

Proof. We show the desired properties in the solution to the more relaxed problem were incentive compatibility

$$\Pi(\theta) - \Pi(\underline{\theta}) = -\int_{\underline{\theta}}^{\theta} C_\theta(q(\theta'), \theta')\, d\theta'$$
$$q(\theta) : \text{increasing}$$

is replaced with the following

$$\Pi(\theta) - \Pi(\underline{\theta}) \leq -\int_{\underline{\theta}}^{\theta} C_\theta(q(\theta'), \theta')\, d\theta', \forall \theta \leq \theta^* \tag{26}$$
$$\Pi(\overline{\theta}) - \Pi(\theta) \geq -\int_{\theta}^{\overline{\theta}} C_\theta(q(\theta'), \theta')\, d\theta', \forall \theta \geq \theta^* \tag{27}$$
$$q(\theta) : \text{increasing}$$

As we will show, in the solution of this more relaxed programing problem, the above inequalities are binding.

In order to show the claim, we first show that if the majorization inequality binds for some value of $\theta' \leq \theta^*$, then it must be binding for all values of $\theta \leq \theta'$. The argument is similar to that of proof of Proposition 5. Hence, we skip the details and describe in brief. In particular, suppose that there exists an interval $I = [\theta_1, \theta_2]$ with $\theta_2 \leq \theta^*$, where majorization is binding at θ_1 and θ_2 while it is slack for all values of $\theta \in (\theta_1, \theta_2)$. Then the exact same steps as in proof of Proposition 5 lead to a contradiction.

Next, we show that for any $\theta > \theta^*$, the majorization constraint is slack. First, note that the incentive constraints (27) must be binding for all values of $\theta > \theta^*$. Suppose to the contrary that the incentive constraint is slack for an interval, I, of θ's above θ^*. Then let $\hat{\theta}$ be the mid-point of I and consider the following perturbation:

$$\delta\overline{q}(\theta) = \begin{cases} \varepsilon & \theta \in I, \theta \leq \hat{\theta} \\ -\varepsilon' & \theta \in I, \theta > \hat{\theta} \\ 0 & \theta \notin I \end{cases}$$

where $\varepsilon, \varepsilon' > 0$ and $\varepsilon\left[F\left(\hat{\theta}\right) - F(\min I)\right] - \varepsilon'\left[F(\max I) - F\left(\hat{\theta}\right)\right] = 0$. This is incentive compatible for a small enough values of $\varepsilon, \varepsilon'$ since incentive compatibility is slack over I. Moreover, since signaled qualities are being allocated to lower θ's, the perturbed allocation satisfies majorization. Hence, it must increase the value of the objective since $\lambda(\theta)$ is strictly decreasing for values of $\theta \geq \theta^*$.

Now, suppose to the contrary that for some $\theta' > \theta^*$, majorization is binding. Given that we have argued that (27) is binding for all values $\theta \geq \theta^*$ and since $\int_{\theta}^{\overline{\theta}} C_\theta\left(q\left(\hat{\theta}\right), \hat{\theta}\right) d\hat{\theta}$ is continuous in θ, then $\Pi(\theta)$ must be continuous over $\left[\theta^*, \overline{\theta}\right]$. Moreover, since majorization is binding at θ', we must have

$$\int_{\theta'}^{\theta} \left[\overline{q}\left(\hat{\theta}\right) - q\left(\hat{\theta}\right)\right] dF\left(\hat{\theta}\right) \geq 0, \theta > \theta'$$
$$\int_{\theta}^{\theta'} \left[\overline{q}\left(\hat{\theta}\right) - q\left(\hat{\theta}\right)\right] dF\left(\hat{\theta}\right) \leq 0, \theta < \theta'$$

Dividing the top inequality by $F(\theta) - F(\theta')$ and bottom one by $F(\theta') - F(\theta)$ and taking limit as θ tends to θ', using l'hÃŽpital's rule, we have[27]

$$\overline{q}(\theta') \geq q(\theta')$$
$$\overline{q}(\theta'-) \leq q(\theta'-)$$

These imply that

$$\begin{aligned}\Pi(\theta') = \overline{q}(\theta'-) - C(q(\theta'-), \theta') &\leq q(\theta'-) - C(q(\theta'-), \theta') \\ &\leq q^{FB}(\theta') - C\left(q^{FB}(\theta'), \theta'\right)\end{aligned}$$

Now, we show that given this property there is an alternative allocation that improves the objective in the relaxed problem. In particular, consider the solution of the problem

$$\max_{\overline{q}(\theta), q(\theta)} \int_{\theta'}^{\overline{\theta}} \Pi(\theta) \lambda(\theta) dF(\theta)$$

subject to

$$\Pi'(\theta) = -C_\theta(q(\theta), \theta)$$
$$\Pi(\theta) = \overline{q}(\theta) - C(q(\theta), \theta)$$
$$q(\theta): \text{ monotone}$$
$$\int_{\theta'}^{\overline{\theta}} \Pi(\theta) dF(\theta) = \int_{\theta'}^{\overline{\theta}} [q(\theta) - C(q(\theta), \theta)] dF(\theta)$$

Let the solution to the above be referred to as $\{q_r(\theta), \overline{q}_r(\theta)\}_{\theta \in \left[\theta', \overline{\theta}\right]}$. As we have shown in the proof of Proposition 4, the solution of the above problem satisfies $C_q \leq 1$ with a strict inequality for a positive measure of types. This would imply that in the solution of the above problem $\Pi_r(\theta') > q^{FB}(\theta') - C\left(q^{FB}(\theta'), \theta'\right)$; otherwise, the first best allocation would deliver a higher objective. This also implies that given the contradiction assumption, $\int_{\theta'}^{\overline{\theta}} \Pi(\theta) \lambda(\theta) dF(\theta) < \int_{\theta'}^{\overline{\theta}} \Pi_r(\theta) \lambda(\theta) dF(\theta)$.

Now, we consider the following allocation: $\{q(\theta), \overline{q}(\theta)\}_{\theta < \theta'}, \{q_r(\theta), \overline{q}_r(\theta)\}_{\theta \geq \theta'}$. This obviously satisfies incentive compatibility for values of $\theta \leq \theta^*$. Moreover, we have

$$\Pi_r\left(\overline{\theta}\right) - \Pi(\theta') > \Pi_r\left(\overline{\theta}\right) - \Pi_r(\theta') = -\int_{\theta'}^{\overline{\theta}} C_\theta\left(q_r\left(\hat{\theta}\right), \hat{\theta}\right) d\hat{\theta}$$

[27] $q(\theta-)$ is the left limit of $q(\cdot)$ at θ.

and thus the allocation satisfies incentive compatibility constraint 27 and improves the objective. We, thus have a contradiction.

So far, we have established that there must exist a threshold $\tilde{\theta} \leq \theta^*$ below which majorization constraint is binding while above it the majorization constraint is slack. Since below $\tilde{\theta}$ majorization is binding, we must have that $\overline{q}(\theta) = q(\theta)$ for all values of $\theta \leq \tilde{\theta}$. As a result, $\Pi\left(\tilde{\theta}\right) \leq q^{FB}\left(\tilde{\theta}\right) - C\left(q^{FB}\left(\tilde{\theta}\right), \tilde{\theta}\right)$. Note that incentive compatibility combined with $\overline{q}(\theta) = q(\theta)$ implies that there exists a threshold $\hat{\theta}$ such that for $\theta \leq \hat{\theta}$, $q(\theta) = q^{FB}(\theta)$ and $q(\theta) = q^{FB}\left(\hat{\theta}\right) = q\left(\tilde{\theta}\right)$ for all $\theta \in \left[\hat{\theta}, \tilde{\theta}\right]$. □

A.8 Proof of Proposition 8

Proof. In order to prove this result, we first show the following lemma:

Lemma 3. *For any subinterval $[x_0, x_2)$ of $[0,1]$ and any $x_1 \in [x_0, x_2)$, consider signaled quality functions $\overline{x}_s(x)$ and $\overline{x}_p(x)$ defined over $[x_0, x_2)$ as follows:*

$$\overline{x}_s(x) = \begin{cases} \frac{\int_{x_0}^{x_1} xh(x)dx}{\int_{x_0}^{x_1} h(x)dx} & x \in [x_0, x_1) \\ \frac{\int_{x_1}^{x_2} xh(x)dx}{\int_{x_1}^{x_2} h(x)dx} & x \in [x_1, x_2) \end{cases}$$

$$\overline{x}_p(x) = \frac{\int_{x_0}^{x_2} xh(x)\,dx}{\int_{x_0}^{x_2} h(x)\,dx}, \forall x \in [x_0, x_2)$$

Then, $\int_{x_0}^{x_2} \Gamma(x)\,\overline{x}_p(x)\,h(x)\,dx \geq \int_{x_0}^{x_2} \Gamma(x)\,\overline{x}_s(x)\,h(x)\,dx$ if and only if

$$\frac{\int_{x_0}^{x_1} \Gamma(x)\,h(x)\,dx}{\int_{x_0}^{x_1} h(x)\,dx} \geq \frac{\int_{x_1}^{x_2} \Gamma(x)\,h(x)\,dx}{\int_{x_1}^{x_2} h(x)\,dx}. \tag{28}$$

Proof. Let us define $a_1 = \frac{\int_{x_0}^{x_1} \Gamma(x)h(x)dx}{\int_{x_0}^{x_1} h(x)dx}$, $a_2 = \frac{\int_{x_1}^{x_2} \Gamma(x)h(x)dx}{\int_{x_1}^{x_2} h(x)dx}$, $b_1 = \frac{\int_{x_0}^{x_1} xh(x)dx}{\int_{x_0}^{x_1} h(x)dx}$, $b_2 = \frac{\int_{x_1}^{x_2} xh(x)dx}{\int_{x_1}^{x_2} h(x)dx}$ and $\alpha = \int_{x_0}^{x_1} h(x)\,dx$, $\beta = \int_{x_1}^{x_2} h(x)\,dx$. Since $b_1 < b_2$, by Chebyshev's sum inequality – see Hardy et al. (1934), Theorem 43 – $a_1 \geq a_2$ if and only if

$$(\alpha + \beta)(\alpha a_1 b_1 + \beta a_2 b_2) \leq (\alpha a_1 + \beta a_2)(\alpha b_1 + \beta b_2) \tag{29}$$

We have

$$\begin{aligned} \alpha a_1 b_1 + \beta a_2 b_2 &= \int_{x_0}^{x_1} \Gamma(x)\,h(x)\,dx \frac{\int_{x_0}^{x_1} xh(x)\,dx}{\int_{x_0}^{x_1} h(x)\,dx} + \int_{x_1}^{x_2} \Gamma(x)\,h(x)\,dx \frac{\int_{x_1}^{x_2} xh(x)\,dx}{\int_{x_1}^{x_2} h(x)\,dx} \\ &= \int_{x_0}^{x_2} \Gamma(x)\,\overline{x}_s(x)\,h(x)\,dx \\ \alpha a_1 + \beta a_2 &= \int_{x_0}^{x_1} \Gamma(x)\,h(x)\,dx + \int_{x_1}^{x_2} \Gamma(x)\,h(x)\,dx = \int_{x_0}^{x_2} \Gamma(x)\,h(x)\,dx \\ \alpha b_1 + \beta b_2 &= \int_{x_0}^{x_1} xh(x)\,dx + \int_{x_1}^{x_2} xh(x)\,dx = \int_{x_0}^{x_2} xh(x)\,dx \end{aligned}$$

Thus (29) becomes

$$\int_{x_0}^{x_2} h(x)\,dx \int_{x_0}^{x_2} \Gamma(x)\,\overline{x}_s(x)\,h(x)\,dx \leq \int_{x_0}^{x_2} \Gamma(x)\,h(x)\,dx \int_{x_0}^{x_2} xh(x)\,dx$$

or

$$\int_{x_0}^{x_2} \Gamma(x)\,\overline{x}_s(x)\,h(x)\,dx \leq \int_{x_0}^{x_2} \Gamma(x)\,\overline{x}_p(x)\,h(x)\,dx$$

This proves the claim. □

In words, the above lemma implies that if the solution to (29), involves two consecutive pooling intervals, then it must be that average value of $\Gamma(x)$ is lower at the lower pooling interval. A similar argument to that of Proposition 7 shows that for the solution to (P'), there must exist a collection of half-intervals $\{K_\alpha\}_{\alpha\in A}$ where each half-interval involves either pooling, i.e., $\overline{x}(\cdot)$ is constant for $x \in K_\alpha$, or it is fully separating, i.e., $\overline{x}(x) = x$ for all $x \in K_\alpha$. We use the above lemma to show that if two half-intervals K_α and K_β which are both pooling and $\sup K_\alpha = \min K_\beta$, then $\frac{\int_{K_\alpha}\Gamma(x)h(x)dx}{\int_{K_\alpha}h(x)dx} = \frac{\int_{K_\beta}\Gamma(x)h(x)dx}{\int_{K_\beta}h(x)dx}$ and thus, we can simply replace K_α, K_β with their union $K_\alpha \cup K_\beta$ – this is also implied by Lemma (3):

Lemma 4. *Let $\overline{x}(x)$ be a solution to (P') and suppose that $x_0 < x_1 < x_2$ exists such that $\overline{x}(\cdot)$ is constant over $[x_0, x_1)$ and $[x_1, x_2)$; with $\overline{x}(x_0-) < \overline{x}(x_0) = \overline{x}(x_1-) < \overline{x}(x_2-) < \overline{x}(x_2)$. Then, $\frac{\int_{x_0}^{x_1}\Gamma(x)h(x)dx}{\int_{x_0}^{x_1}h(x)dx} = \frac{\int_{x_1}^{x_2}\Gamma(x)h(x)dx}{\int_{x_1}^{x_2}h(x)dx}$.*

Proof. Suppose to the contrary that $\frac{\int_{x_0}^{x_1}\Gamma(x)h(x)dx}{\int_{x_0}^{x_1}h(x)dx} \neq \frac{\int_{x_1}^{x_2}\Gamma(x)h(x)dx}{\int_{x_1}^{x_2}h(x)dx}$. If $\frac{\int_{x_0}^{x_1}\Gamma(x)h(x)dx}{\int_{x_0}^{x_1}h(x)dx} > \frac{\int_{x_1}^{x_2}\Gamma(x)h(x)dx}{\int_{x_1}^{x_2}h(x)dx}$, then an alternative signaled quality function $\overline{x}_a(x)$ would deliver a higher payoff:

$$\overline{x}_a(x) = \begin{cases} \frac{\int_{x_0}^{x_2} xh(x)dx}{\int_{x_0}^{x_2} h(x)dx} & x \in [x_0, x_2) \\ \overline{x}(x) & \text{otherwise} \end{cases}$$

That $\overline{x}_a(x)$ delivers a higher objective relative to $\overline{x}(\cdot)$ is a direct result of Lemma 3. This is because $\overline{x}_a$ pools all values of x in $[x_0, x_2)$ while $\overline{x}$ separates the interval $[x_0, x_1)$ from $[x_1, x_2)$. This is a contradiction as $\overline{x}(\cdot)$ was assumed to be optimal.

Now, suppose that $\frac{\int_{x_0}^{x_1}\Gamma(x)h(x)dx}{\int_{x_0}^{x_1}h(x)dx} < \frac{\int_{x_1}^{x_2}\Gamma(x)h(x)dx}{\int_{x_1}^{x_2}h(x)dx}$. Since $\overline{x}(\cdot)$ is optimal and it pools values of x in each interval $[x_0, x_1)$ and $[x_1, x_2)$, an argument similar to above can be used to show that the following must hold

$$\frac{\int_{x_1}^{x}\Gamma(x')\,h(x')\,dx'}{\int_{x_1}^{x} h(x')\,dx'} \geq \frac{\int_{x}^{x_2}\Gamma(x')\,h(x')\,dx'}{\int_{x}^{x_2} h(x')\,dx'}, \forall x \in [x_1, x_2) \tag{30}$$

$$\frac{\int_{x_0}^{x}\Gamma(x')\,h(x')\,dx'}{\int_{x_0}^{x} h(x')\,dx'} \geq \frac{\int_{x}^{x_1}\Gamma(x')\,h(x')\,dx'}{\int_{x}^{x_1} h(x')\,dx'}, \forall x \in [x_0, x_1) \tag{31}$$

This is because if for some x, the above are reversed – for example (30), we can use Lemma 3 to show that separating $[x_1, x)$ and $[x, x_2)$ would increase the value of the objective which is a contradiction. Now, consider (30). We can take limit of x as it converges to x_1. Using l'HÃŽpital's rule, we have the following

$$\Gamma(x_1) \geq \frac{\int_{x_1}^{x_2} \Gamma(x)\, h(x)\, dx}{\int_{x_1}^{x_2} h(x)\, dx}$$

Similarly, by taking the limit as x converges to x_2, we have

$$\frac{\int_{x_1}^{x_2} \Gamma(x)\, h(x)\, dx}{\int_{x_1}^{x_2} h(x)\, dx} \geq \Gamma(x_2)$$

Hence,

$$\Gamma(x_1) \geq \frac{\int_{x_1}^{x_2} \Gamma(x)\, h(x)\, dx}{\int_{x_1}^{x_2} h(x)\, dx} \geq \Gamma(x_2)$$

Using a similar argument, we have

$$\Gamma(x_0) \geq \frac{\int_{x_0}^{x_1} \Gamma(x)\, h(x)\, dx}{\int_{x_0}^{x_1} h(x)\, dx} \geq \Gamma(x_1)$$

This is in contradiction with our initial assumption of $\frac{\int_{x_0}^{x_1} \Gamma(x)h(x)dx}{\int_{x_0}^{x_1} h(x)dx} < \frac{\int_{x_1}^{x_2} \Gamma(x)h(x)dx}{\int_{x_1}^{x_2} h(x)dx}$ as the above inequalities imply that $\frac{\int_{x_0}^{x_1} \Gamma(x)h(x)dx}{\int_{x_0}^{x_1} h(x)dx} \geq \Gamma(x_1) \geq \frac{\int_{x_1}^{x_2} \Gamma(x)h(x)dx}{\int_{x_1}^{x_2} h(x)dx}$. This completes the proof. □

The above lemma establishes that for the solution of (P') we can focus our attention on signaled quality functions that are alternating partitions since we can assume that there are not consecutive pooling intervals at the optimum. The next lemma establishes that there can only be k partitions by showing that if there is a fully revealing interval at the optimum $\Gamma(x)$ must be increasing over this interval:

Lemma 5. *Let $\overline{x}(\cdot)$ be a solution to (P') and suppose that $x_1 < x_2$ exist such that $\overline{x}(x) = x, \forall x \in [x_1, x_2)$ and $\overline{x}(x_1-) < x_1$ and $\overline{x}(x_2) > x_2$. Then, $\Gamma(x)$ is weakly increasing over $[x_1, x_2)$.*

Proof. Suppose to the contrary that the gain function is not increasing over $[x_1, x_2)$. Then there must exist a subinterval $[x_3, x_4)$ of $[x_1, x_2)$ wherein $\Gamma(x)$ is strictly decreasing – this is because $\Gamma'(x)$ is a continuous function. Now, let us consider the following alternative signaled quality function

$$\overline{x}_a(x) = \begin{cases} \frac{\int_{x_3}^{x_4} xh(x)dx}{\int_{x_3}^{x_4} h(x)dx} & x \in [x_3, x_4) \\ \overline{x}(x) & \text{otherwise} \end{cases}$$

Note that by Chebyshev sum (integral) inequality since $\Gamma(x)$ is strictly decreasing over $[x_3, x_4)$ and $h(\cdot)$ is full support,

$$\int_{x_3}^{x_4} xh(x)\,dx \int_{x_3}^{x_4} \Gamma(x)\,h(x)\,dx > \int_{x_3}^{x_4} \Gamma(x)\,xh(x)\,dx \int_{x_3}^{x_4} h(x)\,dx$$

or

$$\frac{\int_{x_3}^{x_4} xh(x)\,dx}{\int_{x_3}^{x_4} h(x)\,dx} \int_{x_3}^{x_4} \Gamma(x)\,h(x)\,dx > \int_{x_3}^{x_4} x\Gamma(x)\,h(x)\,dx.$$

Hence,

$$\int_0^1 \Gamma(x)\,\overline{x}_a(x)\,h(x)\,dx - \int_0^1 \Gamma(x)\,\overline{x}(x)\,h(x)\,dx =$$
$$\frac{\int_{x_3}^{x_4} xh(x)\,dx}{\int_{x_3}^{x_4} h(x)\,dx} \int_{x_3}^{x_4} \Gamma(x)\,h(x)\,dx - \int_{x_3}^{x_4} x\Gamma(x)\,h(x)\,dx > 0$$

which is a contradiction since $\overline{x}(\cdot)$ is optimal. □

Lemmas 4 and 5 establish our claim. By Lemma 4, solution of (P') is an alternating partition – alternating between pooling and separating – and by Lemma 5, its separating parts are a subset of increasing parts of $\Gamma(x)$. Since $\Gamma'(x)$ switches sign k times, we must have that at the optimum there are at most k intervals. □

A.9 Proof of Proposition 9

Proof. To prove the claim, we provide a characterization of the properties of the gain function and use . Note that the gain function is given by

$$\Gamma(x) = \frac{g(x|q_1)}{h(x)} \left(1 + \gamma_1 \frac{g_q(x|q_1)}{g(x|q_1)} + \gamma_2 \frac{g_q(x|q_2)}{g(x|q_2)} \frac{g(x|q_2)}{g(x|q_1)}\right) - 1$$

Before, characterizing properties of the gain function, we prove that γ_2 is positive. To do so, we consider an alternative planning problem

$$\max \int_0^1 \overline{x}(x)\,g(x|q_1)\,dx - C(q_1, \theta_1) \tag{P2}$$

subject to

$$\int_0^1 \overline{x}(x)\,g_q(x|q_1)\,dx = C(q_1, \theta_1) \tag{32}$$
$$\int_0^1 \overline{x}(x)\,g_q(x|q_2)\,dx \geq C(q_2, \theta_2) \tag{33}$$

together with majorization and monotonicity constraints. By Kuhn-Tucker's conditions, the lagrange multiplier associated with the inequality incentive constraint (33) must be either positive or zero (in case the constraint is slack). Thus, in order to show that γ_2 is positive, it is sufficient to show that in (P2), (33) is binding.

Proof that (33) is binding. Suppose that (33) is slack. In this case, we can show that γ_1 is positive. To see that we consider a planning problem without the constraint (33) and with an inequality version of (32). It is straightforward to see that in this problem this constraint must be binding since if slack the gain function $\Gamma(x) = g(x|q_1)/h(x)$ is decreasing and thus optimal $\overline{x}(x)$ is full pooling which then violates the inequality incentive constraint. Hence, $\gamma_1 \geq 0$.

Therefore, the gain function is given by $\Gamma(x) = g(x|q_1)/h(x)\left[1 + \gamma_1 \frac{g_q(x|q_1)}{g(x|q_1)}\right]$. Defining $z(x) = g(x|q_2)/g(x|q_1)$, we can write

$$\Gamma(x) = \hat{\Gamma}(z(x)) = \frac{1}{f_1 + f_2 z(x)}\left[1 + \gamma_1 \phi(z(x))\right]$$

where $\phi(\cdot)$ is defined in Assumption

Since $z(\cdot)$ is increasing in x, determining the sign of $\Gamma'(x)$ is equivalent to that of $\hat{\Gamma}'(z)$. We have

$$\begin{aligned}
\hat{\Gamma}'(z) &= -f_2 \frac{1 + \gamma_1 \phi(z)}{(f_1 + f_2 z)^2} + \frac{\gamma_1 \phi'(z)}{(f_1 + f_2 z)} \\
\left((f_1 + f_2 z)^2 \hat{\Gamma}'(z)\right)' &= -f_2 \gamma_1 \phi'(z) + f_2 \gamma_1 \phi'(z) + (f_1 + f_2 z)\gamma_1 \phi''(z) \\
&= (f_1 + f_2 z)\gamma_1 \phi''(z) \leq 0
\end{aligned}$$

Since by Assumption 3 $\phi''(z) \leq 0$, this implies that $(f_1 + f_2 z)^2 \hat{\Gamma}'(z)$ is decreasing. Thus, there must exist z_1 where $\hat{\Gamma}'(z) \leq 0$ for values of $z \leq z_1$ and $\hat{\Gamma}'(z) \geq 0$ for values of $z \geq z_1$. It is possible that $z_1 = \underline{z} = \min_{x\in[0,1]} g(x|q_2)/g(x|q_1)$ or $z_1 = \bar{z} = \max_{x\in[0,1]} g(x|q_2)/g(x|q_1)$. Since $\Gamma(x)$ must have the same property, using Proposition 8 there must exist a cutoff x_1 so that below x_1, optimal $\overline{x}(\cdot)$ is fully revealing while above x_1 is pooling. Now given that optimal $\overline{x}(x)$ has this shape when (32) is slack for $i = 2$, consider an infinitesimal increase in q_2 accompanied by a uniform increase in $\overline{x}(\cdot)$ for all values of x so that the change in $\overline{x}(x)$, $\delta\overline{x}(x) = f_2 \delta q_2$. Since (33) is slack, this perturbed allocation keeps it satisfied. Moreover, since $\delta\overline{x}(x)$ is constant this perturbation keeps (32) unchanged. Finally, majorization is satisfied since the perturbed $\overline{x}(\cdot)$ is equal to x only at one point and is higher than x for values of x below this point. This leads to the desired contradiction as this perturbation increases the payoff of type 1 sellers.

Having proven that γ_2 is positive, we can use Assumption 3 to characterize the optimal rating system. Note that in this case, the gain function is given by

$$\Gamma(x) = \hat{\Gamma}(z(x)) = \frac{1}{f_1 + f_2 z(x)}\left[1 + \gamma_1 \phi(z(x)) + \gamma_2 \psi(z(x))\right]$$

We have

$$\begin{aligned}\hat{\Gamma}'(z) =& - f_2 \frac{1+\gamma_1\phi(z)+\gamma_2\psi(z)}{(f_1+f_2 z)^2} \\ &+ \frac{\gamma_1\phi'(z)+\gamma_2\psi'(z)}{f_1+f_2 z} \\ (f_1+f_2 z)^2\hat{\Gamma}'(z) =& - f_2\left[1+\gamma_1\phi(z)+\gamma_2\psi(z)\right] \\ &+ (f_1+f_2 z)\left[\gamma_1\phi'(z)+\gamma_2\psi'(z)\right] \\ \left((f_1+f_2 z)^2\hat{\Gamma}'(z)\right)' =& (f_1+f_2 z)\left[\gamma_1\phi''(z)+\gamma_2\psi''(z)\right]\end{aligned}$$

There are two possibilities:

1. The multiplier γ_1 is negative. In this case, $\gamma_1\phi''(z)+\gamma_2\psi''(z)\geq 0$ by Assumption (3) which implies that either $\Gamma'(x)$ is always positive or negative for low values of x and positive for high values. Hence, by Proposition (8) the solution is of the form

$$\overline{x}(x) = \begin{cases}\mathbb{E}_h\left[x|x<x_2\right] & x<x_2\\ x & x\geq x_2\end{cases}$$

   where $\mathbb{E}_h[\cdot|\cdot]$ is conditional expectation according to the distribution $H(\cdot)$ and $x_2\in[0,1]$. The above proves the claim since in this case $x_1=0$.

2. The multiplier γ_1 is positive. In this case, the sign of $\gamma_1\phi''(z)+\gamma_2\psi''(z)$ cannot be determined. However, part 3 of Assumption 3 implies that it is negative for low values of z while it is positive for high values of z. Hence, either $\hat{\Gamma}'(\underline{z})<0$ which means that $\hat{\Gamma}'(z)$ (and as a result $\Gamma'(x)$)only switches sign once, or $\hat{\Gamma}'(\underline{z})>0$ which means that $\hat{\Gamma}'(z)$ at most changes sign twice. By Proposition 3, all of these cases lead to the optimal rating structure stated in the Proposition. This concludes the proof.

□

B An Algorithm for Construction of Signals

Here, we provide a construction algorithm when the distribution of θ is discrete. The algorithm illustrates that a combination of a rather small class of rating systems, those that simply pool qualities together, can always implement a vector of signaled qualities. Before describing the algorithm, we define two classes of signals; for convenience, we work with measures over posterior beliefs.

1. **Interval pooled system**: For any two indices $k<l$, an interval pooled signal, represented by $\sigma^{k\to l}\in\Delta(\Delta(\Theta))$, is one in which all qualities $q_l,\cdots,q_k$ send the same

signal while the qualities of all other types are fully revealed. Formally,

$$\sigma^{k\to l}\left(\{\mathbf{e}_i\}\right) = f_i, \forall i < k, i > l$$
$$\sigma^{k\to l}\left(\left\{\frac{f_k}{f_k+\cdots+f_l}\mathbf{e}_k+\cdots+\frac{f_l}{f_l+\cdots+f_k}\mathbf{e}_l\right\}\right) = f_k+\cdots+f_l; \text{otherwise},$$

where $\mathbf{e}_i \in \Delta(\Theta)$ is a vector that is 1 in its i-th element and 0 otherwise.

2. **Two-point pooled system**: For any two indices $k < l$, a two-point pooled signal, represented by $\sigma^{k,l} \in \Delta(\Delta(\Theta))$, is one in which qualities q_l and q_k send the same signal while the qualities of all other types are fully revealed. Formally,

$$\sigma^{k,l}\left(\{\mathbf{e}_i\}\right) = f_i, \forall i \neq k, l$$
$$\sigma^{k,l}\left(\left\{\frac{f_k}{f_k+f_l}\mathbf{e}_k+\frac{f_l}{f_k+f_l}\mathbf{e}_l\right\}\right) = f_l+f_k.$$

We also refer to the fully informative signal as $\tau^{FI} \in \Delta(\Delta(\Theta))$ with $\tau^{FI}(\{\mathbf{e}_i\}) = f_i$. Now consider the vectors of signaled and true qualities, $\overline{\mathbf{q}} \neq \mathbf{q}$, such that $\mathbf{q} \succcurlyeq_F \overline{\mathbf{q}}$. Then the following algorithm can be used to construct the rating system that implements the signaled quality $\overline{\mathbf{q}}$:

Algorithm 1. *Start by letting* $\mathbf{r} = \mathbf{q}$. *Let* l *and* k *be defined as follows:*

$$k = \arg\min_i \overline{q}_i > r_i$$

and

$$l = \arg\min_{i>k} \overline{q}_i < r_i.$$

1. *If for all values of* $j \in \{k,\cdots,l-1\}$, $\overline{q}_j > r_j$, *let* $\hat{\lambda}$ *be defined as the highest value of* $\lambda < 1$ *such that*

$$\lambda r_j + (1-\lambda)\frac{f_k r_k+\cdots+f_l r_l}{f_k+\cdots+f_l} \geq \overline{q}_j, \forall j \in \{k,\cdots,l-1\}$$
$$\lambda r_l + (1-\lambda)\frac{f_k r_k+\cdots+f_l r_l}{f_k+\cdots+f_l} \leq \overline{q}_l$$

with at least one equality. Using this value of $\hat{\lambda}$, *we construct* $\hat{\tau} = \left(1-\hat{\lambda}\right)\sigma^{k\to l} + \hat{\lambda}\tau^{FI}$ *and* $\tilde{\mathbf{r}} = diag(\mathbf{f})^{-1}\int_{\Delta(\Theta)} \boldsymbol{\mu}\boldsymbol{\mu}^T \mathbf{r}\, d\hat{\tau}$.

- *If for some value of* $j \in \{k,\cdots,l-1\}$, $\overline{q}_j = r_j$, *let* $k' \in \{k+1,\cdots,l-1\}$ *satisfy* $\overline{q}_{k'} > r_{k'}$ *and* $\overline{q}_{k'+1} = r_{k'+1}$. *In addition, let* $\hat{\lambda}$ *be the highest value of* $\lambda < 1$ *that satisfies*

$$\lambda r_{k'} + (1-\lambda)\frac{f_{k'} r_{k'}+f_l r_l}{f_{k'}+f_l} \leq \overline{q}_{k'}$$
$$\lambda r_l + (1-\lambda)\frac{f_{k'} r_{k'}+f_l r_l}{f_{k'}+f_l} \geq \overline{q}_l,$$

with at least one of the above holding with equality. Using this value of $\hat{\lambda}$, *we construct* $\hat{\tau} = \left(1 - \hat{\lambda}\right) \sigma^{k,l} + \hat{\lambda} \tau^{FI}$ *and* $\tilde{\mathbf{r}} = diag\left(\mathbf{f}\right)^{-1} \int_{\Delta(\Theta)} \boldsymbol{\mu} \boldsymbol{\mu}^T \mathbf{r} d\hat{\tau}$.

- *If* $\tilde{\mathbf{r}} \neq \overline{\mathbf{q}}$, *repeat the above steps replacing* $\mathbf{r}$ *with* $\tilde{\mathbf{r}}$.

The proof that this algorithm works uses the fact that the set $\mathcal{S}$ defined in (6) is convex. In each step of the above algorithm, the number of elements of $\mathbf{r}$ and $\overline{\mathbf{q}}$ that are different shrinks by at least l. As a result, the repetition of this procedure, while keeping majorization intact, gives us the rating system that maps $\mathbf{q}$ into $\overline{\mathbf{q}}$. As the algorithm shows, the constructed signal is derived from a repeated application of interval and two-point pooled signals. While we do not show an equivalent result for continuous distributions, one can use the above algorithm in an approximate form by approximating continuous distributions with discrete ones.

B.1 Proof of Algorithm (1)

Proof. For any $\mathbf{q} \succcurlyeq_F \overline{\mathbf{q}}$, define l and k as follows:

$$k = \arg\min_i \overline{q}_i > q_i$$

and

$$l = \arg\min_{i > k} \overline{q}_i < q_i$$

There are two possibilities:

1. For all values of $j \in \{k, \cdots, l-1\}$ we have that $\overline{q}_j > q_j$.
   In this case, we define the following signal

   $$\hat{\tau} = \lambda \cdot \tau^{FI} + (1 - \lambda) \cdot \sigma^{k \to l}$$

   and its associated matrix

   $$\begin{aligned} \mathbf{A} &= \lambda \int \boldsymbol{\mu} \boldsymbol{\mu}^T d\tau^{FI} + (1 - \lambda) \int \boldsymbol{\mu} \boldsymbol{\mu}^T d\sigma^{k \to l} \\ &= \lambda I + (1 - \lambda) \Sigma^{k \to l} \end{aligned}$$

   Therefore

   $$\hat{\mathbf{r}}_i = (\mathbf{A}\mathbf{q})_i = \begin{cases} q_i & i < k, i > l \\ \lambda q_i + (1 - \lambda) & i = k, \cdots, l \end{cases}$$

   Note that because of our choice of this transformation, the elements of $\hat{\mathbf{p}}$ are monotone. This is because

   $$\cdots \geq q_{l+1} \geq q_l \geq \lambda q_l + (1 - \lambda) \overline{q}_{k \to l} \geq \cdots \geq \lambda q_{k+1} + (1 - \lambda) \overline{q}_{k \to l} \geq \lambda q_k + (1 - \lambda) \overline{q}_{k \to l} > q_k \geq q_{k-1} \geq$$

   We can find the highest value of $\lambda \in [0, 1]$ such that one of the following equalities hold

   $$\lambda q_i + (1 - \lambda) \overline{q}_{k \to l} = \overline{q}_i, i = k, \cdots, l$$

Such a value must exist since $\overline{q}_i \geq q_i, \forall k < i < l$ and $\overline{q}_l < q_l$. Let this value of λ be called $\hat{\lambda}$. From the definition of λ, it implies that the following inequalities must hold

$$\forall j = k, \cdots, l-1, \hat{\lambda} q_j + \left(1 - \hat{\lambda}\right) \overline{q}_{k \to l} \leq \overline{q}_j$$
$$\hat{\lambda} q_l + \left(1 - \hat{\lambda}\right) \overline{q}_{k \to l} \geq \overline{q}_l$$

Note further that by construction

$$f_k \hat{r}_k + \cdots + f_l \hat{r}_l = f_k q_k + \cdots + f_l q_l$$

We then have

$$\sum_{j=1}^{i} f_j \hat{r}_j = \begin{cases} \sum_{j=1}^{i} f_j q_j & i < k \\ \sum_{j=1}^{k-1} f_j q_j + \sum_{j=k}^{i} f_j \left[\hat{\lambda} q_j + \left(1 - \hat{\lambda}\right) \overline{q}_{k \to l}\right] & i = k, \cdots, l-1 \\ \sum_{j=1}^{i} f_j q_j & i = l \end{cases}$$

By the above inequalities the above obviously mean that

$$\sum_{j=1}^{i} f_j \hat{r}_j \leq \sum_{j=1}^{i} f_j \overline{q}_j$$

This implies that in this constructed signal $\hat{\mathbf{r}} \succcurlyeq_F \overline{\mathbf{q}}$. Moreover, obviously we must have that $\mathbf{q} \succcurlyeq_F \hat{\mathbf{r}}$ – from Lemma 2.

2. There exists $j \in \{k+1, \cdots, l-1\}$ such that $\overline{q}_j = q_j$.
   In this case, let k' satisfy $\overline{q}_{k'} > q_{k'}$ and $\overline{q}_{k'+1} = q_{k'+1}$. This must necessarily exist since $\overline{q}_k > q_k$. Now let $\hat{\lambda}$ be the highest value of $\lambda \in [0,1]$ that satisfies

$$\lambda q_{k'} + (1-\lambda) \frac{f_{k'} q_{k'} + f_l q_l}{f_{k'} + f_l} \leq \overline{q}_{k'}$$
$$\lambda q_l + (1-\lambda) \frac{f_{k'} q_{k'} + f_l q_l}{f_{k'} + f_l} \geq \overline{q}_l$$

with at least one of the above holding with equality. Then we define

$$\hat{\tau} = \lambda \cdot \tau^{FI} + (1-\lambda) \sigma^{k',l}$$

Then obviously the resulting $\hat{\mathbf{r}}$ is monotone in its elements and with an argument similar to above it F-majorizes $\overline{\mathbf{q}}$ and is F-majorized by $\mathbf{q}$.

□