47802 - Macroeconomics 1

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Final Exam

Problem 1. A Model of On-the-Job Search Consider the following version of the McCall search model that allows for on-the-job-search. In particular, suppose that time is continuous and workers - employed or unemployed discount the future at rate ρ. A worker that has a job separates from her job at rate η. An unemployed worker finds a job at rate λ_0 - the wage is then drawn from a distribution $F(w)$ which is continuous. Moreover, a worker who has a job gets an opportunity to draw a new wage offer at rate λ_1 - again from the same distribution $F(w)$. The worker then decides whether to accept this new offer or not.

As usual, assume that the flow value of unemployment is b; the workers are risk-neutral and cannot save.

a. Let $V(w)$ be the value of a job at hand that pays the wage w . Let U be the value of unemployment. Write the Bellman equation for U and $V(w)$.

Solution. The value functions are given by

$$\rho V(w) = w + \lambda_1 \int \max\{V(w') - V(w), 0\} dF(w') + \eta(U - V(w)) \tag{1}$$

$$\rho U = b + \lambda_0 \int \max\{V(w) - U, 0\} dF(w) \tag{2}$$

b. Calculate the reservation wage, w^*, for an unemployed worker as a function of the value function $V(w)$. Under what conditions, $w^* < b$ holds? Provide an intuition for this result.

Solution. For w^*, we must have that $U = V(w^*)$. Subtracting, (2) from (1) evaluated at w^*, we have

$$w^* = b + (\lambda_0 - \lambda_1) \int \max\{V(w) - V(w^*), 0\} dF(w)$$

When $\lambda_0 < \lambda_1$, the above shows that $w^* < b$. Intuitively, when $\lambda_1 > \lambda_0$, accepting a job gives a better option value to a worker than being unemployed - a higher probability of a new draw which always increases the value for a worker. This implies that an unemployed worker is willing to take a job at a wage lower than unemployment benefit in order to obtain this better option value.

c. What is the reservation wage for an employed worker?

Solution. Since $V(w)$ is an increasing function of w, the reservation wage of an employed worker is w - the wage of the job she is currently employed in.

d. How does an increase in unemployment benefits, b, affect the job-to-job turnovers? Show your result and explain intuitively.

Solution. We have

$$\begin{aligned}(\rho+\eta)V(w) &= w+\lambda_1\int_w [V(w')-V(w)]\,dF(w')+\eta U \\ (\rho+\eta)V'(w) &= 1-\lambda_1 V'(w)(1-F(w)) \\ V'(w) &= \frac{1}{\rho+\eta+\lambda_1(1-F(w))}\end{aligned}$$

Therefore,

$$\begin{aligned}w^* &= b+(\lambda_0-\lambda_1)\int_{w^*}[V(w)-V(w^*)]\,dF(w) \\ &= b-(\lambda_0-\lambda_1)\int_{w^*}[V(w)-V(w^*)]\,d(1-F(w)) \\ &= b+(\lambda_0-\lambda_1)\int_{w^*}(1-F(w))\,V'(w)\,dw \\ &= b+(\lambda_0-\lambda_1)\int_{w^*}\frac{1-F(w)}{\rho+\eta+\lambda_1(1-F(w))}dw\end{aligned}$$

We have

$$\begin{aligned}\frac{\partial w^*}{\partial b} &= 1-(\lambda_0-\lambda_1)\frac{\partial w^*}{\partial b}\frac{1-F(w^*)}{\rho+\eta+\lambda_1(1-F(w^*))} \\ \frac{\partial w^*}{\partial b}\left[1+\frac{(\lambda_0-\lambda_1)(1-F(w^*))}{\rho+\eta+\lambda_1(1-F(w^*))}\right] &= 1 \\ \frac{\partial w^*}{\partial b}\left[\frac{\rho+\eta+\lambda_0(1-F(w^*))}{\rho+\eta+\lambda_1(1-F(w^*))}\right] &= 1\end{aligned}$$

This implies that $\frac{\partial w^*}{\partial b}>0$. A rise in the reservation wage leads to a lower turnover as less people accept jobs and therefore less people move between jobs as well.

Problem 2. A Stochastic AK economy
Consider an economy in which production is done only with capital and output is given by

$$Y_t = A_t K_t$$

where A_t is an i.i.d. stochastic process and is distributed according to the c.d.f. $F(A)$. Capital depreciates at rate δ. Suppose that the economy exhibits a representative agent and that preferences of such hypothetical agent is given by

$$\sum_{t=0}^{\infty} \beta^t u(C_t)$$

where $u(\cdot)$ is a CRRA utility function given by

$$u(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma} & \gamma \neq 1 \\ \log c & \gamma = 1 \end{cases}$$

a. Define a Competitive Equilibrium for this economy assuming that trading occurs in sequential asset markets.

Solution. A competitive equilibrium for this economy is a sequence of allocations $\Big\{C_t(s^t), X_t(s^t), K_{t+1}(s^t), B$
as well as prices $\{p_t(s^t), r_t(s^t), q_{t+1}(s^{t+1})\}$ where $s_t = A_t$ and:

1. Households solve:

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t)\, u(C_t(s^t))$$

subject to

$$p_t(s^t)\left[C_t(s^t) + X_t(s^t)\right] + \sum_{s^{t+1} \succeq s^t} q_{t+1}(s^{t+1})\, B_{t+1}(s^{t+1}) \leq r_t(s^t)\, K_t(s^{t-1}) + p_t(s^t)\, B_t(s^t)$$
$$(1-\delta) K_t(s^{t-1}) + X_t(s^t) = K_{t+1}(s^t)$$

2. Firms solve

$$\max_k p_t(s^t)\, A_t(s^t)\, k - r_t(s^t)\, k$$

3. Markets clear

$$B_t(s^t) = 0$$
$$K_t^f(s^t) = K_t(s^{t-1})$$
$$C_t(s^t) + X_t(s^t) = A_t(s^t)\, K_t(s^{t-1})$$

Note that if A has a continuous distribution, the above really should be re-written with integrals over s^t instead of sums.

b. State the First Welfare Theorem and use it to formulate a competitive equilibrium as a solution to a planning problem.

Solution. According to the FWT, a competitive equilibrium must be pareto optimal. As a result, it must solve the following planning problem

$$\max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi\left(s^t\right) u\left(C_t\left(s^t\right)\right)$$

subject to

$$C_t\left(s^t\right) + K_{t+1}\left(s^t\right) = A_t\left(s^t\right) K_t\left(s^{t-1}\right) + (1-\delta) K_t\left(s^{t-1}\right)$$

c. Formulate the planning problem problem recursively. You are required to use only one state variable.

Solution. Bellman equation is given by

$$V(Y) = \max u(C) + \beta \int V\left(\left(A' + 1 - \delta\right) K'\right) dF\left(A'\right)$$

subject to

$$C + K' = Y$$

d. Solve the functional equation associated with this recursive problem and find the value function and policy functions using a guess and verify method. What assumptions on the fundamentals should be made so that the solution to this problem exists and is unique?

Solution. We guess that $V(Y) = B\frac{Y^{1-\gamma}}{1-\gamma}$. With this guess, the optimization in the FE above becomes

$$\max_{K'} \frac{\left(Y - K'\right)^{1-\gamma}}{1-\gamma} + \beta B \int \frac{\left(\left(A' + 1 - \delta\right) K'\right)^{1-\gamma}}{1-\gamma} dF\left(A'\right)$$

If we let $x = K'/Y$, then solving the above optimization is equivalent to

$$\max_{x} \frac{(1-x)^{1-\gamma}}{1-\gamma} + \beta B \int \frac{\left(\left(A' + 1 - \delta\right) x\right)^{1-\gamma}}{1-\gamma} dF\left(A'\right)$$

Taking FOCs, we have

$$\begin{aligned}
(1-x)^{-\gamma} &= \beta B x^{-\gamma} \mathbb{E}\left[(A + 1 - \delta)^{1-\gamma}\right] \\
\left(\frac{x}{1-x}\right)^{\gamma} &= \beta B \mathbb{E}\left[(A + 1 - \delta)^{1-\gamma}\right]
\end{aligned}$$

For each B the above has a unique solution. We can therefore, replacing in the objective in the FE and have

$$\begin{aligned}
v(Y) &= \frac{(C)^{1-\gamma}}{1-\gamma} + \beta B \int \frac{\left(\left(A' + 1 - \delta\right) K'\right)^{1-\gamma}}{1-\gamma} dF\left(A'\right) \\
&= \frac{Y^{1-\gamma}}{1-\gamma}\left[(1-x)^{1-\gamma} + \beta B x^{1-\gamma} \mathbb{E}\left[(A + 1 - \delta)^{1-\gamma}\right]\right] \\
&= \frac{Y^{1-\gamma}}{1-\gamma}\left[(1-x)^{1-\gamma} + x(1-x)^{-\gamma}\right] = \frac{Y^{1-\gamma}}{1-\gamma}(1-x)^{-\gamma} \\
&= \beta B x^{-\gamma} \mathbb{E}\left[(A + 1 - \delta)^{1-\gamma}\right] \frac{Y^{1-\gamma}}{1-\gamma}
\end{aligned}$$

which has the form as the guessed value. If we set the above equal to $B\frac{Y^{1-\gamma}}{1-\gamma}$, we have

$$\beta x^{-\gamma}\mathbb{E}\left[(A+1-\delta)^{1-\gamma}\right]=1\Rightarrow x=\left(\beta\mathbb{E}\left[(A+1-\delta)^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}$$

and

$$B=(1-x)^{-\gamma}=\left(1-\left(\beta\mathbb{E}\left[(A+1-\delta)^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}\right)^{-\gamma}$$

Therefore, in order for this solution to be meaningful, we must have that

$$\beta\mathbb{E}\left[(A+1-\delta)^{\frac{1}{\gamma}}\right]<1.$$

The policy function has the form

$$K'=xY=x\left(A+1-\delta\right)K$$

e. Using the policy functions calculated above, calculate the average growth rate of this economy - assuming that the data comes from the stationary distribution of the model. The result should be expressed in terms of the fundamentals parameters of the model as well as various moments of the distribution $F(\cdot)$.

Solution. The growth rate of the economy is given by

$$1+g_{t+1}=\frac{A_{t+1}K_{t+1}}{A_tK_t}=\frac{A_{t+1}x\left(A_t+1-\delta\right)K_t}{A_tK_t}=x\frac{A_{t+1}\left(A_t+1-\delta\right)}{A_t}$$

Since shocks are i.i.d., this average growth rate is given by

$$\begin{aligned}1+\overline{g}&=x\mathbb{E}\left[A\right]\mathbb{E}\left[\frac{A+1-\delta}{A}\right]\\&=\left(\beta\mathbb{E}\left[(A+1-\delta)^{1-\gamma}\right]\right)^{\frac{1}{\gamma}}\mathbb{E}\left[A\right]\mathbb{E}\left[\frac{A+1-\delta}{A}\right]\end{aligned}$$

In what follows, assume that $\delta=1$ and that

$$F\left(A\right)=\int_0^A\frac{1}{\sigma x\sqrt{2\pi}}e^{-\frac{\left(\log x+\sigma^2/2-\log\overline{A}\right)^2}{2\sigma^2}}dx$$

That is, A is distributed according to a log-normal distribution whose mean is $\overline{A}$ and its variance is $\left(e^{\sigma^2}-1\right)\overline{A}^2$.

f. What is the average growth rate of the economy? How does growth interact with risk, i.e., how does the average growth rate of the economy depend on σ? Explain your answer intuitively.

Solution. From above, we have

$$1 + \overline{g} = \left(\beta \mathbb{E}\left[A^{1-\gamma}\right]\right)^{\frac{1}{\gamma}} \mathbb{E}\left[A\right]$$

Note that $\log A \sim \mathcal{N}\left(\log \overline{A} - \frac{\sigma^2}{2}, \sigma^2\right)$. Therefore,

$$\mathbb{E}\left[A^{1-\gamma}\right] = \mathbb{E}\left[e^{(1-\gamma)\log A}\right] = e^{(1-\gamma)\log \overline{A} + \frac{(1-\gamma)\gamma}{2}\sigma^2}$$

Therefore,

$$1 + \overline{g} = \beta^{\frac{1}{\gamma}} \overline{A}^{\frac{1}{\gamma}} e^{\frac{(1-\gamma)\sigma^2}{2}}$$

Therefore, an increase in risk σ^2 leads to an increase in growth when $\gamma < 1$ and a decrease in growth when $\gamma > 1$. The intuition for this result can be seen by rewriting the Euler equation as follows:

$$\begin{aligned} u'(C_t) &= \beta \mathbb{E}_t\left[A_{t+1} u'(C_{t+1})\right] \\ &= \beta\left\{\mathbb{E}_t A_{t+1} \mathbb{E}_t u'(C_{t+1}) + Cov_t\left(A_{t+1}, u'(C_{t+1})\right)\right\} \end{aligned}$$

An increase in risk, has two effects: 1. it increases the average marginal utility $\mathbb{E}_t u'(C_{t+1})$. This is because marginal utility is a convex function of consumption and an increase in risk raises its average value - it increases the probability of tail events. The consumers would like to invest more to insure themselves against the risk of such events as a precaution. 2. it decreases the correlation between marginal utility and returns; that is, it makes it more negative - usual risk-aversion. When $\gamma < 1$, the precautionary motive dominates while when $\gamma > 1$, the risk-aversion motive dominates. Thus for low values of γ, an increase in risk leads to higher investment and higher growth while for high values of γ an increase in risk leads to low investment and growth.

g. Calculate the price of a risk-free real bond, P_t^f. What is the risk-free rate in this economy.

Solution. The price of a risk-free bond is given by

$$\begin{aligned} P_t^f &= \beta \mathbb{E}_t \frac{u'(C_{t+1})}{u'(C_t)} \\ &= \beta \mathbb{E}_t \frac{\left((1-x) x A_{t+1} A_t K_t\right)^{-\gamma}}{\left((1-x) A_t K_t\right)^{-\gamma}} \\ &= \beta x^{-\gamma} \mathbb{E}_t A_{t+1}^{-\gamma} \\ &= \beta \left(\beta \mathbb{E}\left[A^{1-\gamma}\right]\right)^{-\frac{\gamma}{\gamma}} \mathbb{E}\left[A^{-\gamma}\right] \\ &= \frac{\mathbb{E}\left[A^{-\gamma}\right]}{\mathbb{E}\left[A^{1-\gamma}\right]} \end{aligned}$$

The return on the asset is given by

$$R_t^f = \frac{1}{P_t^f} = \frac{\mathbb{E}\left[A^{1-\gamma}\right]}{\mathbb{E}\left[A^{-\gamma}\right]}$$

h. Calculate the price, P_t, of holding a firm that owns the physical capital and invests from the output that it produces and show that $P_t = K_t$; Hint: This is an asset with a dividend process given by $D_t = A_t K_t - K_{t+1}$. What is the return on such an asset?

Solution. We have

$$\begin{aligned}
P_t &= \beta \mathbb{E}_t \left[\frac{u'(C_{t+1})}{u'(C_t)} [D_{t+1} + P_{t+1}] \right] \\
&= \beta \mathbb{E}_t \left[\sum_{s=t+1}^{\infty} \beta^{s-t-1} \frac{C_s^{-\gamma}}{C_t^{-\gamma}} D_s \right] \\
&= \beta \mathbb{E}_t \left[\sum_{s=t+1}^{\infty} \beta^{s-t-1} \frac{C_s^{-\gamma}}{C_t^{-\gamma}} (A_s K_s - K_{s+1}) \right] \\
&= \beta \mathbb{E}_t \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} A_{t+1} K_{t+1} \\
&+ \beta \mathbb{E}_t \left[\sum_{s=t+2}^{\infty} \beta^{s-t-1} \frac{C_s^{-\gamma}}{C_t^{-\gamma}} \left(A_s K_s - \frac{1}{\beta} \frac{C_{s-1}^{-\gamma}}{C_s^{-\gamma}} K_s \right) \right] \\
&= \beta \mathbb{E}_t \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} A_{t+1} K_{t+1} \\
&+ \beta \frac{1}{C_t^{-\gamma}} \mathbb{E}_t \left[\sum_{s=t+2}^{\infty} \beta^{s-t-2} C_{s-1}^{-\gamma} \left(\beta \frac{C_s^{-\gamma}}{C_{s-1}^{-\gamma}} A_s K_s - K_s \right) \right] \\
&= \beta \mathbb{E}_t \frac{C_{t+1}^{-\gamma}}{C_t^{-\gamma}} A_{t+1} K_{t+1} + \beta \frac{1}{C_t^{-\gamma}} \mathbb{E}_t \left[\sum_{s=t+2}^{\infty} \beta^{s-t-2} C_{s-1}^{-\gamma} K_s \left(\beta \frac{C_s^{-\gamma}}{C_{s-1}^{-\gamma}} A_s - 1 \right) \right]
\end{aligned}$$

By the Euler equation, we have

$$\begin{aligned}
\mathbb{E}_{s-1} \left[\beta \frac{C_s^{-\gamma}}{C_{s-1}^{-\gamma}} A_s \right] = 1 \Rightarrow \mathbb{E}_{s-1} \left[\beta \frac{C_s^{-\gamma}}{C_{s-1}^{-\gamma}} A_s - 1 \right] = 0 \\
\Rightarrow \mathbb{E}_t \left[\beta \frac{C_s^{-\gamma}}{C_{s-1}^{-\gamma}} A_s - 1 \right] = 0
\end{aligned}$$

where the last equality follows from law of iterated expectations. We therefore have

$$\begin{aligned}
P_t &= K_{t+1} + \beta \frac{1}{C_t^{-\gamma}} \mathbb{E}_t \left[\sum_{s=t+2}^{\infty} \beta^{s-t-2} C_{s-1}^{-\gamma} K_s \mathbb{E}_{s-1} \left(\beta \frac{C_s^{-\gamma}}{C_{s-1}^{-\gamma}} A_s - 1 \right) \right] \\
&= K_{t+1}
\end{aligned}$$

which completes the claim.

The return on the capital is given by

$$R_{t+1}^k = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{A_{t+1} K_{t+1} - K_{t+2} + K_{t+2}}{K_{t+1}} = A_{t+1}$$

i. Calculate the equity premium as the difference between average return on physical capital and the risk-free rate. How does this depend on risk-aversion, γ, and variance of the shocks, σ. Explain intuitively.

Solution. The equity premium is defined as

$$\begin{aligned}
\mathbb{E}\left[R_t^k - R_t\right] &= \mathbb{E}A - \frac{\mathbb{E}\left[A^{1-\gamma}\right]}{\mathbb{E}\left[A^{-\gamma}\right]} \\
&= \frac{\mathbb{E}A\mathbb{E}\left[A^{-\gamma}\right] - \mathbb{E}\left[A^{1-\gamma}\right]}{\mathbb{E}\left[A^{-\gamma}\right]} \\
&= \frac{\overline{A}e^{-\gamma\left(\log\overline{A}-\frac{\sigma^2}{2}\right)+\frac{\gamma^2\sigma^2}{2}} - e^{(1-\gamma)\left(\log\overline{A}-\frac{\sigma^2}{2}\right)+\frac{(1-\gamma)^2\sigma^2}{2}}}{e^{-\gamma\left(\log\overline{A}-\frac{\sigma^2}{2}\right)+\frac{\gamma^2\sigma^2}{2}}} \\
&= \frac{\overline{A}^{1-\gamma}e^{\frac{\gamma(1+\gamma)\sigma^2}{2}} - \overline{A}^{1-\gamma}e^{-\frac{(1-\gamma)\gamma\sigma^2}{2}}}{\overline{A}^{1-\gamma}e^{\frac{\gamma(1+\gamma)\sigma^2}{2}}} \\
&= \overline{A}\left[1 - e^{-\gamma\sigma^2}\right]
\end{aligned}$$

This is an increasing function of γ and σ^2. As σ rises, the risk inherent in holding physical capital increases and as a result investor will require a higher rate of return. The same logic holds for γ; as households become more risk-averse they demand a higher average rate of return in order to hold the capital stock.