Problem 1

See the solutions to PS#1 from last year.

Problem 2

Part (a)

- We first prove that L^p is a vector space.

  Let $f, g \in L^p$. Since $f : [0,1] \to \mathbb{R}$, $g : [0,1] \to \mathbb{R}$, $f + g : [0,1] \to \mathbb{R}$.

  Note that $|f+g|^p \leq |2\max\{f,g\}|^p = 2^p|\max\{f,g\}|^p \leq 2^p(|f|^p + |g|^p)$.

  Then $\int_0^1 |f(x)+g(x)|^p dx \leq 2^p \int_0^1 |f(x)+g(x)|^p dx + 2^p \int_0^1 |g(x)|^p dx < \infty$.

  Thus $f + g \in L^p$.

  Now, let $f \in L^p$ and $\alpha \in \mathbb{R}$. Since $f : [0,1] \to \mathbb{R}$, $\alpha f : [0,1] \to \mathbb{R}$. $\int_0^1 |\alpha f(x)|^p dx = \int_0^1 |\alpha|^p \int_0^1 |f(x)|^p dx \leq \infty$. Thus $\alpha f \in L^p$.

  Therefore, L^p is a vector space.

- We now prove that $(\int_0^1 |f(x)|^p dx)^{(1/p)} \forall f \in L^p$ is a norm.

  1. Show (a) $||f||_p \geq 0$ (b) $||f||_p = 0 \iff f = 0$

     (a) $\forall x \in [0,1]$ $|f(x)| \geq 0 \Rightarrow |f(x)|^p \geq 0$ $\forall x \in [0,1] \Rightarrow \int_0^1 |f(x)|^p dx \geq 0$
     $\Rightarrow (\int_0^1 |f(x)|^p dx)^{(1/p)} \geq 0 \Rightarrow ||f||_p \geq 0$

     (b) $||f||_p = 0 \Rightarrow (\int_0^1 |f(x)|^p dx)^{(1/p)} = 0 \Rightarrow \int_0^1 |f(x)|^p dx = 0 \Rightarrow |f(x)| = 0$ $\forall x \in [0,1]$
     $\Rightarrow f(x) = 0$ $\forall x \in [0,1] \Rightarrow f = 0$
     $f = 0 \Rightarrow f(x) = 0$ $\forall x \in [0,1] \Rightarrow |f(x)| = 0$ $\forall x \in [0,1] \Rightarrow |f(x)|^p = 0$ $\forall x \in [0,1]$
     $\Rightarrow \int_0^1 |f(x)|^p dx = 0 \Rightarrow (\int_0^1 |f(x)|^p dx)^{(1/p)} = 0 \Rightarrow ||f||_p = 0$.

  2. $||\alpha f||_p = (\int_0^1 |\alpha f(x)|^p dx)^{(1/p)} = (\int_0^1 |\alpha|^p |f(x)|^p dx)^{(1/p)}$
     $= (|\alpha|^p \int_0^1 |f(x)|^p dx)^{(1/p)} = |\alpha|(\int_0^1 |f(x)|^p dx)^{(1/p)} = |\alpha| ||f||_p$

  3. $||f+g||_p \leq ||f||_p + ||g||_p$

     $$||f+g||_p^p = \int_0^1 |f(x)+g(x)|^p dx = \int_0^1 |f(x)+g(x)||f(x)+g(x)|^{(p-1)} dx$$
     $$\leq \int_0^1 (|f(x)| + |g(x)|)|f(x)+g(x)|^{(p-1)} dx$$
     $$= \int_0^1 |f(x)||f(x)+g(x)|^{(p-1)} dx + \int_0^1 |g(x)||f(x)+g(x)|^{(p-1)} dx$$
     $$\leq ((\int_0^1 |f(x)|^p dx)^{(1/p)} + (\int_0^1 |g(x)|^p dx)^{(1/p)})(\int_0^1 |f(x)+g(x)|^{(p-1)(p/p-1)} dx)^{(1-1/p)}$$

(Hölder's inequality)

$= (||f||_p + ||g||_p)\frac{||f+g||_p^p}{||f+g||_p}$

Therefore, $||f+g||_p \leq ||f||_p + ||g||_p$.

Part (b)

If $\{f_k\}$ is Cauchy in L^∞, then for every $m \in \mathbb{N}$ there exists an integer $n \in \mathbb{N}$ such that we have

$$|f_j(x) - f_k(x)| \geq 1/m \text{ for all } j, k \geq n \text{ and } x \in N^c_{j,k,m} \quad (1)$$

where $N_{j,k,m}$ is a null set.

Let $N = \bigcup_{j,k,m\in\mathbb{N}} N_{j,k,m}$. Then N is a null set, and for every $x \in N^c$ the sequence $\{f_k(x) : k \in \mathbb{N}\}$ is Cauchy in $\mathbb{R}$. We define a measurable function $\{f : X \to \mathbb{R}\}$, unique up to a pointwise a.e. equivalence, by

$$f(x) = \lim_{k\to\infty} f_k(x) \text{ for } x \in N^c$$

Letting $k \to \infty$ in (1), we find that for every $m \in \mathbb{N}$ there exists an integer $n \in \mathbb{N}$ such that

$$|f_j(x) - f(x)| \leq 1/m \text{ for } j \geq n \text{ and } x \in N^c \quad (2)$$

It follows that f is essentially bounded and $f_j \to f$ in L^∞ as $j \to \infty$.

This proves that L^∞ is complete.

Part (c)

Andre (solutions to PS#1, problem 2, part b) has given an example for this part. Here I give an example that illustrates the problem with the integral norm more clearly. Set $a = -1$, $b = 1$. Consider the sequence

$$x_n(t) = \begin{cases} -1 & x \in [-1, -\frac{1}{n}) \\ nt & x \in [-\frac{1}{n}, \frac{1}{n}] \\ 1 & x \in (\frac{1}{n}, 1] \end{cases}$$

Note that each function $x_n(t)$ is continuous. We need to verify that the sequence $\{x_n\}$ is Cauchy. WLOG let $n \leq m$.

$$||x_n - x_m|| = \int_{-1}^{1} |x_n(t) - x_m(t)|dt = \int_{-\frac{1}{n}}^{-\frac{1}{m}} |nt+1|dt + \int_{-\frac{1}{m}}^{\frac{1}{m}} |(n-m)t|dt + \int_{\frac{1}{m}}^{\frac{1}{n}} |nt-1|dt = \frac{1}{n} - \frac{1}{m};\ \forall n, m \in \mathbb{N}$$

$\forall\epsilon > 0$, $\exists N(\epsilon)\, s.t. \forall n, m > N(\epsilon)$, $||x_n - x_m|| < \epsilon$. (To see this, just take $N(\epsilon) > \frac{1}{\epsilon}$). Therefore, $\{x_n\}$ is Cauchy.

Next, note that this sequence converges pointwise and **w.r.t the integral norm** to

$$x(t) = \begin{cases} -1 & x \in [-1, 0) \\ 0 & x = 0 \\ 1 & x = (0, 1] \end{cases}$$

To see the convergence under the integral norm, consider

$$||x_n - x|| = \int_{-1}^{1} |x_n(t) - x(t)|dt =$$
$$\int_{-1}^{-\frac{1}{n}} |-1 - (-1)|dt + \int_{-\frac{1}{n}}^{0} |nt - (-1)|dt + \int_{0}^{\frac{1}{n}} |nt - 1|dt + \int_{\frac{1}{n}}^{1} |1 - 1|dt = \frac{1}{n} \to 0.$$

But the function $x(t)$ is not continuous. Hence although the Cauchy sequence converges, its limit does not belong to the space S.

This example does not work under the sup norm $||x(t)|| = |\sup x(t)|$ because the sequence is not Cauchy. Assume WLOG that $n > m$. We have

$$|x_n(t) - x_m(t)| = \begin{cases} 0 & t \in [-1, -\frac{1}{n}) \\ |nt + 1| & t \in [-\frac{1}{n}, -\frac{1}{m}) \\ |(n - m)t| & t \in [-\frac{1}{n}, -\frac{1}{m}) \\ |nt - 1| & [-\frac{1}{n}, -\frac{1}{m}) \\ 0 & [-\frac{1}{n}, -\frac{1}{m}] \end{cases}$$

Therefore, $||x_n - x_m|| = |\sup x_n(t) - x_m(t)| = \frac{n-m}{m}$. Now take $m = 2n$, we get $||x_n - x_m|| = \frac{1}{2}$. Which shows that the sequence is not Cauchy with respect to the sup norm.

Problem 3

Let us define T to be

$$(Tx)(t) = \int_{0}^{t} g(z)(1 - x(z)^2)dz$$

We will use Blackwell's sufficient conditions for a contraction to show this is an contraction. Then by the contraction mapping theorem, T will have exactly one fixed point satisfying the differential equation.

(i) Monotonicity: Let $x(t), y(t)$ be bounded continuous functions mapping $[0,1]$ to $\mathbb{R}_+$ with $x(t) \geq y(t) \quad \forall t \in [0,1]$. We see that

$$\begin{aligned}
(Tx)(t) - (Ty)(t) &= \int_0^t g(z)(1 - x(z)^2)dz - \int_0^t g(z)(1 - y(z)^2)dz \\
&= \int_0^t g(z)x(z)^2 dz - \int_0^t g(z)y(z)^2 dz \\
&= \int_0^t g(z)(x(z)^2 - y(z)^2)dz \\
&\geq 0
\end{aligned}$$

Therefore $(Tx)(t) \geq (Ty)(t) \quad \forall t \in [0,1]$.

(ii) Discounting: Let $(x+a)(t) = x(t) + a$. Then

$$\begin{aligned}
[T(x+a)](t) &= \int_0^t g(z)(1 - (x(z)+a)^2)dz \\
&= \int_0^t g(z)(1 - (x(z)^2 + 2ax(z) + a^2))dz \\
&= \int_0^t g(z)(1 - x(z)^2)dz + \int_0^t g(z)(-2ax(z) - a^2)dz \\
&= (Tx)(t) + \int_0^t g(z)(-2ax(z) - a^2)dz \\
&\leq (Tx)(t) + \frac{1}{3}\int_0^t (-2ax(z) - a^2)dz \quad \text{since } g(t) \in [0, 1/3] \quad \forall t \in [0,1] \\
&= (Tx)(t) + \frac{1}{3}a\int_0^t (-2x(z) - a)dz \\
&\leq (Tx)(t) + \frac{1}{3}a \quad \text{since } x(0) = 0 \text{ and } x : [0,1] \to \mathbb{R}_+
\end{aligned}$$

Thus T is a contraction mapping and the CMT applies, ensuring a unique x satisfying the given equation.

Problem 4

Part (a)

Note first of all that in a sequential equilibrium, the budget constraint is homogeneous of degree 0 in prices and we can normalize $p_{c,t} = 1$ (or if you prefer, we can divide by $p_{c,t}$ and write our other prices as ratios). Hence we can write the household's problem (suppressing notation for j) as

$$\max_{\{c_t,x_t,k_{t+1},a_{t+1},l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t,l_t) \ s.t.$$
$$c_t + p_{x,t}x_t + q_t a_t \leq w_t(e_t - l_t) + r_t k_t + a_t \ \forall \ t$$
$$x_t = k_{t+1} - (1-\delta)k_t$$
$$a_t \geq -A \ \forall \ t$$
$$a_0, k_0 \text{ given} \tag{3}$$

For feasibility we still need the goods, labor, and capital markets to clear, but now we have the additional constraint that

$$\sum_j a_t^j = 0$$

Intuitively, for every borrower we need a lender, so that in the aggregate household debt sums to zero. Recall Walras' law: if every market but one clears, then all markets clear. Hence if the goods markets clear, then the lending market has to clear, which means that borrowing = lending.

Part (b)

A competitive equilibrium is allocations for the household $\{c_t^j, x_t^j, l_t^j, k_t^j, a_t^j\}_{j=1,\ldots,J}^{t=0,\ldots,\infty}$ and firm $\{y_t^f, k_t^f, n_t^f\}_{f=c,x}^{t=0,\ldots,\infty}$ and prices $\{q_t, p_{x,t}, r_t, w_t\}_{t=0}^{\infty}$ such that

- Households solve (3)
- The representative firms (simplifying here) solve

$$\max_{y_t,k_t,l_t} p_{t,f} y_t^f - w_t n_t^f - r_t k_t^f \ s.t.$$
$$y_t^f = F^f(k_t^f, n_t^f)$$

 for $f \in \{c, x\}$

- Markets clear:

$$\sum_j c_t^j = y_t^c$$
$$\sum_j x_t^j = y_t^x$$
$$\sum_j (e_t^j - l_t^j) = \sum_f n_t^f$$
$$\sum_j k_t^j = \sum_f k_t^f$$
$$\sum_j a_t^j = 0 \qquad \forall\ i$$

- Profits are consistent with firm behavior (here excluded since profits are zero at the CE)

Part (c)

We want to show that the Arrow-Debreu budget constraint is equivalent to the budget constraint with sequential markets. The AD budget constraint is

$$\sum_{t=0}^{\infty} (p_{c,t}c_t + p_{x,t}x_t) \leq \sum_{t=0}^{\infty} (w_t(e_t - l_t) + r_t k_t)$$
$$\implies \sum_{t=0}^{\infty} (p_{c,t}c_t + p_{x,t}k_{t+1}) \leq \sum_{t=0}^{\infty} (w_t(e_t - l_t) + r_t k_t + p_{x,t}(1-\delta)k_t)$$

Consider the FOC for this problem:

$$\beta^t u_c(c_t, l_t) = p_{c,t}\lambda$$
$$\beta^t u_l(c_t, l_t) = w_t\lambda$$
$$p_{x,t} = (r_{t+1} + p_{x,t+1}[1-\delta])$$

Note that from the firm's problem, we have $r_t = p_{x,t}F_k(k_t^x, n_t^x)$. Therefore we can rewrite and combine the FOC to show that, at an interior CE,

$$u_c(c_t, l_t) = \frac{p_{c,t}}{p_{c,t+1}}\beta u_c(c_{t+1}, l_{t+1})$$
$$u_c(c_t, l_t) = \frac{p_{c,t}}{w_t}u_l(c_t, l_t)$$
$$\frac{p_{x,t}}{p_{x,t+1}} = (F_k(k_{t+1}^x, n_{t+1}^x) + 1 - \delta)$$
$$= \frac{p_{c,t}}{p_{c,t+1}}$$

where the last equality follows from the fact that both firms pay the same rental rate for capital (otherwise one firm would be unable to rent capital!). Now turning our attention to the sequential markets economy and taking the FOC,

$$\begin{aligned}
\beta^t u_c(c_t, l_t) &= \lambda_t \\
\beta^t u_l(c_t, l_t) &= w_t \lambda_t \\
\lambda_t p_{x,t} &= \lambda_{t+1} \left(r_{t+1} + p_{x,t+1}(1-\delta) \right) \\
\lambda_t q_t &= \lambda_{t+1}
\end{aligned}$$

We can rewrite these conditions as

$$\begin{aligned}
u_c c_t, l_t &= \frac{\lambda_t}{\lambda_{t+1}} \beta u_c(c_{t+1}, n_{t+1}) \\
u_c(c_t, l_t) &= \frac{1}{w_t} u_l(c_t, l_t) \\
\frac{\lambda_t}{\lambda_{t+1}} &= \frac{p_{x,t+1}}{p_{x,t}} (F_k(k^x_{t+1}, n^x_{t+1}) + 1 - \delta) \\
&= (F_k(k^x_{t+1}, n^x_{t+1}) + 1 - \delta) \qquad \text{(why?)} \\
&= \frac{1}{q_t}
\end{aligned}$$

But then we have shown that

$$\begin{aligned}
\text{AD} : u_c(c_t, l_t) &= \beta u_c(c_{t+1}, l_{t+1})(F_k(k^x_{t+1}, n^x_{t+1}) + 1 - \delta) \\
\text{Sequential} : u_c(c_t, l_t) &= \beta u_c(c_{t+1}, l_{t+1})(F_k(k^x_{t+1}, n^x_{t+1}) + 1 - \delta)
\end{aligned}$$

If we perform the same exercise with the *intratemporal* Euler equation (i.e. the labor-leisure trade-off), we will obtain the same result: the Euler equations for these two economies are identical. Moreover, we know that because the firms' problem is the same in both economies, the firms' allocations will be the same; and we have therefore shown that the CE in the Arrow-Debreu economy is equivalent to the CE in the sequential markets economy, and the relationship between prices in these two economies is $\frac{1}{q_t} = r_{t+1} + 1 - \delta$.

Part (d)

From above, and supposing we hadn't normalized prices, we would have

$$\begin{aligned} p_{x,t} &= \frac{\lambda_{t+1}}{\lambda_t} \left(r_{t+1} + p_{x,t+1}(1-\delta) \right) \\ &= \frac{q_t}{p_{c,t+1}} \left(r_{t+1} + p_{x,t+1}(1-\delta) \right) \\ &= p_{c,t} \left(r_{t+1} + p_{x,t+1}(1-\delta) \right) \\ &= r_{t+1} + p_{x,t+1}(1-\delta) \end{aligned}$$

or rearranging,

$$r_{t+1} - p_{x,t} + p_{x,t+1}(1-\delta) = 0$$

In words, the rental value of capital is offset by the capital gains loss. If this equality didn't hold - for instance, if $r_{t+1} > p_{x,t} - p_{x,t+1}(1-\delta)$ - then the household could purchase a unit of capital today for $p_{x,t}$ and tomorrow receive $p_{x,t+1}(1-\delta) + r_{t+1}$, yielding positive profits after repaying $p_{x,t}$. To see how r_t relates to q_t, also from above (and with normalized prices) we have

$$\frac{1}{q_{t-1}} = r_{t+1} + (1-\delta)$$

or the return on bonds (LHS) is equal to the return on capital (RHS). Here as well, if the equality didn't hold then we would have an arbitrage opportunity: sell short bonds and invest, or sell short next-period capital and buy bonds, depending upon which return is lower.

Problem 5

Part (a)

A competitive equilibrium is allocations for the households $\{c_i\}_{i=1}^N$ and firms $\{K_i, L_i, Q_{i1}, ..., Q_{iN}\}_{i=1}^N$ and prices $\{r, w, p_i\}_{i=1}^N$ such that

- Households solve

$$\begin{aligned} \max \ & \sum_{i=1}^N \alpha_i \log c_i \ s.t. \\ & \sum_{i=1}^N p_i c_i \le rk + wl \end{aligned}$$

  where $k = \bar{K}/I$ and $l = \bar{L}/I$.

- Firms solve

$$\max\ p_iY_i - rK_i - wL_i - \sum_{j=1}^{N} p_jQ_{ij}\ s.t.$$

$$Y_i = A_iK_i^{\alpha}L_i^{\gamma-\alpha}\prod_{j=1}^{N}Q_{ij}^{(1-\gamma)\omega_{ij}}$$

- Markets clear:

$$\bar{C}_i + \sum_{j=1}^{N} Q_{ji} = Y_i\ \forall\ i$$

$$\bar{K} = \sum_{j=1}^{N} K_j$$

$$\bar{L} = \sum_{j=1}^{N} L_j$$

Part(b)

Taking the FOC of the firm's cost-minimization problem, we get

$$r = \lambda\alpha A_iK_i^{\alpha-1}L_i^{\gamma-\alpha}\prod_{j=1}^{N}Q_{ij}^{(1-\gamma)\omega_{ij}}$$

$$w = \lambda(\gamma-\alpha)A_iK_i^{\alpha}L_i^{\gamma-\alpha-1}\prod_{j=1}^{N}Q_{ij}^{(1-\gamma)\omega_{ij}}$$

$$p_j = \lambda(1-\gamma)\omega_{ij}A_iK_i^{\alpha}L_i^{\gamma-\alpha}Q_{ij}^{(1-\gamma)\omega_{ij}-1}\prod_{k\neq j}Q_{ik}^{(1-\gamma)\omega_{ik}}$$

The simplest way to solve this is by multiplying each side by the factor of production for which we took the derivative, and re-obtaining Y:

$$K_i\frac{r}{\alpha} = Y_i\lambda$$

$$L_i\frac{w}{\gamma-\alpha} = Y_i\lambda$$

$$Q_{ij}\frac{p_j}{(1-\gamma)\omega_{ij}} = Y_i\lambda$$

It follows that

$$\begin{aligned}
L_i &= \frac{r(\gamma-\alpha)}{\alpha w}K_i \\
Q_{i,j} &= \frac{r(1-\gamma)\omega_{ij}}{\alpha p_j}K_i \\
\implies Y_i &= A_i\left(\frac{r(\gamma-\alpha)}{\alpha w}\right)^{\gamma-\alpha}\prod_{j=1}^{N}\left(\frac{r(1-\gamma)\omega_{ij}}{\alpha p_j}\right)^{(1-\gamma)\omega_{ij}}K_i \\
\implies K_i &= A_i^{-1}\left(\frac{r(\gamma-\alpha)}{\alpha w}\right)^{\alpha-\gamma}\prod_{j=1}^{N}\left(\frac{r(1-\gamma)\omega_{ij}}{\alpha p_j}\right)^{(\gamma-1)\omega_{ij}}Y_i \\
&= A_i^{-1}\left(\frac{r}{\alpha}\right)^{\alpha-1}\left(\frac{\gamma-\alpha}{w}\right)^{\alpha-\gamma}\prod_{j=1}^{N}\left(\frac{p_j}{(1-\gamma)\omega_{ij}}\right)^{(1-\gamma)\omega_{ij}}Y_i \\
&= P_i^kY_i
\end{aligned}$$

Using the same steps we can show that each input is some constant fraction of Y:

$$\begin{aligned}
L_i &= A_i^{-1}\left(\frac{\alpha}{r}\right)^{-\alpha}\left(\frac{w}{\gamma-\alpha}\right)^{\gamma-\alpha-1}\prod_{j=1}^{N}\left(\frac{p_j}{(1-\gamma)\omega_{ij}}\right)^{(1-\gamma)\omega_{ij}}Y_i = P_i^lY_i \\
Q_{ij} &= A_i^{-1}\left(\frac{\alpha}{r}\right)^{-\alpha}\left(\frac{\gamma-\alpha}{w}\right)^{\alpha-\gamma}\prod_{k\neq j}\left(\frac{p_k}{(1-\gamma)\omega_{ik}}\right)^{(1-\gamma)\omega_{ij}}\left(\frac{p_j}{(1-\gamma)\omega_{ij}}\right)^{(\gamma-1)\omega_{ii}-1}Y_i = P_i^jY_i
\end{aligned}$$

But then, returning to the cost function,

$$\begin{aligned}
C_i(Y) &= rP_i^kY_i + wP_i^lY_i + \sum_{j=1}^{N}p_jP_i^jY_i \\
&= \psi Y_i
\end{aligned}$$

as we wanted to show.

Part (c)

We know that at any solution to the firm's problem, costs are a constant proportion of output - i.e. the proportion of inputs is static and the production function is homogeneous of degree 0. We can therefore consider a 'reduced form' profit function for the firm:

$$\begin{aligned}
\pi(Y_i;p) &= p_iY_i - C(Y_i;p) \\
&= p_iY_i - \psi_iY_i
\end{aligned}$$

where p is the vector of prices. The first-order condition is

$$p_i - \psi_i = 0$$

which states that the firm should produce up to the point that marginal revenue (p_i) equals marginal cost (ψ_i).

Part (d)

We now have the system of equations

$$\begin{aligned} p_i &= rP_i^k + wP_i^l + \sum_{j=1}^N p_j P_i^j \\ &= A_i^{-1}\left(\frac{r}{\alpha}\right)^\alpha \left(\frac{w}{\gamma-\alpha}\right)^{\gamma-\alpha} \prod_{j=1}^N \left(\frac{p_j}{(1-\gamma)\omega_{ij}}\right)^{(1-\gamma)\omega_{ij}} \left[\alpha + (\gamma-\alpha) + \sum_{j=1}^N \omega_{ij}(1-\gamma)\right] \\ &= A_i^{-1}\left(\frac{r}{\alpha}\right)^\alpha \left(\frac{w}{\gamma-\alpha}\right)^{\gamma-\alpha} \left(\frac{1}{1-\gamma}\right)^{1-\gamma} \prod_{j=1}^N \left(\frac{p_j}{\omega_{ij}}\right)^{(1-\gamma)\omega_{ij}} \end{aligned}$$

Taking logs,

$$\ln(p_i) = -\ln A_i + \ln\left(\left(\frac{r}{\alpha}\right)^\alpha \left(\frac{w}{\gamma-\alpha}\right)^{\gamma-\alpha}\right) + \sum_{j=1}^N (1-\gamma)w_{ij}\ln p_j - \sum_{j=1}^N (1-\gamma)w_{ij}\ln((1-\gamma)w_{ij})$$

Switching to matrix notation and letting $\hat{p} = [\log p_1, ..., \log p_N]$, $\hat{A} = [\ln A_1, ..., \ln A_N]$, and Γ correspond to the matrix of input-output elasticities,

$$\begin{aligned} \hat{p} &= -\hat{A} + \vec{Q} + (1-\gamma)\Gamma\hat{p} \\ &= (I - (1-\gamma)\Gamma)^{-1}(-\hat{A} + \vec{Q}) \end{aligned}$$

where $Q_i = \alpha \ln\left(\frac{r}{\alpha}\right) + (\gamma-\alpha)\ln\left(\frac{w}{\gamma-\alpha}\right) - \sum_{j=1}^N (1-\gamma)\omega_{ij}\log([1-\gamma]\omega_{ij})$.

Part (e)

We normalize $w = 1$, in which case from the firm's FOC we have

$$r = \frac{\alpha}{\gamma-\alpha}\frac{L_i}{K_i}$$

The market-clearing conditions for capital and labor are

$$\begin{aligned} \sum_i K_i &= \bar{K} \\ \sum_i L_i &= \bar{L} \end{aligned}$$

and since each firm uses capital and labor in the same proportion, we have

$$r = \frac{\alpha}{\gamma - \alpha} \frac{\bar{L}}{\bar{K}} \tag{4}$$

The price equation $\hat{p}$ now simplifies to

$$\hat{p} = (I - (1-\gamma)\Gamma)^{-1} \left(-\hat{A} + \left(\gamma \ln w + \alpha \ln \left(\frac{\bar{L}}{\bar{K}} \right) \right) \vec{e} - \vec{Q}' \right)$$

where $\vec{e} = \begin{bmatrix} 1 \\ ... \\ 1 \end{bmatrix}$ and $\vec{Q}' = -\gamma \ln(\gamma - \alpha) - \sum_{j=1}^{N} (1-\gamma) w_{ij} \ln((1-\gamma) w_{ij})$, and hence we have solved for all prices as functions of the elasticity terms, productivity, and the aggregate endowments of capital and labor.

To solve for production, we need to turn to the household's problem. The household's FOC are

$$\frac{\alpha_i}{p_i c_i} = \frac{\alpha_j}{p_j c_j}$$
$$\implies c_i = \frac{\alpha_i p_j}{\alpha_j p_i} c_j$$

The household's budget constraint is

$$\sum_i p_i c_i \leq rk + wl$$

which will hold with equality at the optimum. Assuming for simplicity that $\sum \alpha_i = 1$ and solving for demand,

$$\begin{aligned} c_i &= \frac{rk + wl}{p_i} - \sum_{j \neq i} \frac{p_j c_j}{p_i} \\ &= \frac{rk + wl}{p_i} - \sum_{j \neq i} \frac{\alpha_j}{\alpha_i} c_i \\ &= \frac{rk + wl}{p_i} - \frac{1 - \alpha_i}{\alpha_i} c_i \\ &= \frac{\alpha_i}{p_i} (rk + wl) \\ &= \frac{\alpha_i}{p_i} \left(\frac{\gamma}{\gamma - \alpha} \bar{L} \right) \end{aligned}$$

For each good, we have the market clearing equation

$$\sum_i Q_{ij} + c_j = Y_j$$

Substituting back into the market-clearing condition,

$$\sum_i Q_{ij} + \frac{\alpha_i}{p_i}(r\bar{K} + w\bar{L}) = Y_j$$
$$\implies \sum_i P_i^j Y_i + \frac{\alpha_i}{p_i}\left(\frac{\gamma}{\gamma-\alpha}\bar{L}\right) = Y_j$$
$$\implies Y = (I - P')^{-1} d\left(\frac{\gamma}{\gamma-\alpha}\bar{L}\right)$$

where $d_i = \frac{\alpha_i}{p_i}$ and P is the matrix of P_i^j defined earlier.

Part (f)

Nominal GDP is (not assuming $w = 1$)

$$GDP = \sum_i p_i c_i = \sum_i p_i d_i (r\bar{K} + w\bar{L})$$
$$= \sum_i \left(\frac{p_i d_i \gamma w}{\gamma - \alpha}\bar{L}\right)$$
$$= \frac{\gamma w}{\gamma - \alpha}\bar{L}$$

However, in this question we want to solve for the real GDP. Therefore, we need first to find the price index. With the Cobb-Douglas utility function the price index is given by $\prod_{i=1}^N p_i^{\alpha_i}$. We normalize this index to 1, meaning $\log\left(\prod_{i=1}^N p_i^{\alpha_i}\right) = \sum_{i=1}^N \alpha_i \log p_i = \alpha^T \hat{p} = 0$. Now replacing from $\hat{p}$ (part d) we get

$$\alpha^T \hat{p} = \alpha^T (I - (1-\gamma)\Gamma)^{-1}\left(-\hat{A} + \left(\gamma \ln w + \alpha \ln\left(\frac{\bar{L}}{\bar{K}}\right)\right)\vec{e} - \vec{Q}'\right) = 0$$

Next, note that $\Gamma\vec{e} = \vec{e}$, (because $\sum_{j=1}^N w_{ij} = 1, \forall i$), therefore $\vec{e} - (1-\gamma)\Gamma\vec{e} = \gamma\vec{e}$, and we have $(I - (1-\gamma)\Gamma)\vec{e} = \gamma\vec{e}$. Hence, $(I - (1-\gamma)\Gamma)^{-1}\vec{e} = \gamma^{-1}\vec{e}$. We get

$$\alpha^T \hat{p} = \alpha^T (I - (1-\gamma)\Gamma)^{-1}\left(-\hat{A} - \vec{Q}'\right) + \ln w + \alpha \ln\left(\frac{\bar{L}}{\bar{K}}\right) = 0 \rightarrow$$
$$\ln w = \alpha^T (I - (1-\gamma)\Gamma)^{-1}\left(\hat{A} + \vec{Q}'\right) - \alpha \ln\left(\frac{\bar{L}}{\bar{K}}\right) \rightarrow$$
$$w = C\left(\frac{\bar{L}}{\bar{K}}\right)^{-\alpha}$$

Replacing this in nominal GDP gives us the real GDP:

$$GDP_r = \frac{\gamma}{\gamma-\alpha} C \left(\frac{\bar{L}}{\bar{K}}\right)^{-\alpha} \bar{L} = \frac{\gamma}{\gamma-\alpha} C \bar{L}^{1-\alpha}\bar{K}^{\alpha}$$

part (g)

Following the notation in previous parts

$$Y_i = \frac{\alpha_i}{p_i}\frac{\gamma w\bar{L}}{\gamma-\alpha} + (1-\gamma)\sum_{j=1}^{N} w_{ji}\frac{p_j Y_j}{p_i}$$
$$\rightarrow p_i Y_i - (1-\gamma)\sum_{j=1}^{N} w_{ji} p_j Y_j = \alpha_i \frac{\gamma w}{\gamma-\alpha}\bar{L}$$

In matrix form

$$X = [I-(i-\gamma)\Gamma^T]^{-1}\vec{\alpha}\frac{\gamma w}{\gamma-\alpha}\bar{L}$$

where $X = \begin{bmatrix} p_1Y_1 \\ ... \\ p_NY_N \end{bmatrix}$.

To get the real aggregate output we multiply the vector X by $\vec{e}^T$ from left and substitute the normalized wage derived in part f

$$\vec{e}^T X = \gamma^{-1}\vec{e}^T\vec{\alpha}\frac{\gamma w}{\gamma-\alpha}\bar{L} = \frac{w}{\gamma-\alpha}\bar{L} = \frac{C}{\gamma-\alpha}\bar{L}^{1-\alpha}\bar{K}^{\alpha}$$

Now consider a change of $g\%$ in A_i

$$C' = \exp\left[\alpha^T(I-(1-\gamma)\Gamma)^{-1}(\hat{A}+\ln(1+g)\vec{e_i}+\vec{Q}')\right] \quad = C\times\exp[\alpha^T(I-(1-\gamma)\Gamma)^{-1}\ln(1+g)\vec{e_i}]$$

Note that if we assume the productivity of all sectors in the economy is changing, instead of $\vec{e_i}$ we will have the vector $\vec{e}$. In that case $[\alpha^T(I-(1-\gamma)\Gamma)^{-1}\ln(1+g)\vec{e}]$ simplifies to $\gamma^{-1}\ln(1+g)$ (note that I have shown in part f that $\vec{e}$ is an eigenvector of Γ). Hence

$$C' = C\times(1+g)^{\frac{1}{\gamma}} \rightarrow$$
$$\vec{e}^T X = \frac{C(1+g)^{\frac{1}{\gamma}}}{\gamma-\alpha}\bar{L}^{1-\alpha}\bar{K}^{\alpha}$$

Therefore, we see that a $g\%$ change in all A_i's leads to a change of $(\frac{1}{\gamma}g\%)$ in aggregate output. γ here determines the magnitude of the effect. In fact γ captures the extend to which the economy depends on the network structure for production versus mere dependence on capital and labor. Now suppose only 1 of the sectors (sector i) changes. We get

$$[\alpha^T(I-(1-\gamma)\Gamma)^{-1}\vec{e_i}\ln(1+g)] = [\alpha^T(\sum_{k=0}^{\infty}(1-\gamma)^k\Gamma^k)\vec{e_i}\ln(1+g)]$$
$$= \ln(1+g)\left(1+\sum_{j=1}^{N}\alpha_j w_{ji} + \alpha^T(\sum_{k=0}^{\infty}(1-\gamma)^k\Gamma^k)\vec{e_i}\right)$$

I expand the series just to show what kind of terms show up in The $\alpha^T(I-(1-\gamma)\Gamma)^{-1}\vec{e_i}$. For example the second term ($\sum_{j=1}^N \alpha_j w_{ji}$) illustrates that the effect of a shock to sector i depends (among other things) on the number of nodes to which it is directly connected and also the magnitude of this connection denoted by w_{ji}. Generally, a change of $g\%$ in A_i leads to a change in aggregate output wich depends on the terms in the i'th column of $\alpha^T(I-(1-\gamma)\Gamma)^{-1}$ (which captures a measure of centrality for sector i).

Part (h)

Compare the two matrices

$$W_{chain} = \begin{bmatrix} 1 & 0 & ... & ... & 0 & 0 \\ 1 & 0 & ... & ... & 0 & 0 \\ 0 & 1 & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & 0 & 0 \\ 0 & 0 & ... & ... & 1 & 0 \end{bmatrix}$$

and

$$W_{circle} = \begin{bmatrix} 0 & 0 & ... & ... & 0 & 1 \\ 1 & 0 & ... & ... & 0 & 0 \\ 0 & 1 & ... & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & 0 & 0 \\ 0 & 0 & ... & ... & 1 & 0 \end{bmatrix}$$

In the first case, aggregate production function looks like

$$p.Y = p_1 A_1 K_1^\alpha L_1^{\gamma-\alpha} Q_{11}^{1-\gamma} + \sum_{i\geq 2} p_i A_i K_i^\alpha L_i^{\gamma-\alpha} Q_{i,i-1}^{1-\gamma}$$

and in the second case,

$$p.Y = p_1 A_1 K_1^\alpha L_1^{\gamma-\alpha} Q_{1N}^{1-\gamma} + \sum_{i\geq 2} p_i A_i K_i^\alpha L_i^{\gamma-\alpha} Q_{i,i-1}^{1-\gamma}$$

In the first case, sector $i = 1$ is critical; any change in the behavior of this sector will have an 'outsized' effect on aggregate production. In the case of a chain network, other sectors may also be critical: if something (e.g. a productivity shock) affects sector k, then every sector with $i > k$

will also be affected. In the case of a circle network, however, every sector is equally critical. If any link in the circle is affected through a negative productivity shock, it's effects will ripple through the every sector eventually reaching back to the initial sector negatively affecting it again, creating a downward spiral in production. Without something exogenous to correct the negative shock, the negative spiral will continue and never "work its way out of the system" as in the chain case.

Problem 6

In this problem, we develop a model in which education is a commodity that is produced with its own production function, traded at its specific price, and consumed by agents to increase their time endowments. Hence this model requires the commodity space $\{c_t, \ell_t, x_t, k_t, e_t\}_{t=0}^{\infty}$ for a time constraint defined as $\ell \leq e + f(\{e_s\}_{s=0}^{t})$ where $f(\{e_s\}_{s=0}^{t})$ describes the accumulation of education in time. Such accumulation could be a usual LOM for some stock commodity s (an additional commodity in the commodity space): $s_t + 1 = (1-\delta_e)s_t + e_t$.

Under the assumption that education only affects the time constraint of the agent, it has no bearing on her utility. Instead, it affects her budget constraint by increasing the amount of time that can be dedicated to paid labor. Under this assumption, firms' production functions are not directly affected either, except for the fact that there is an additional firm with its own production technology $F^e(k_t^e, n_t^e)$. Note that the problem assumes an identical production technology for consumption and investment good: $F^c(k_t^c, n_t^c) = F^x(k_t^x, n_t^x) = F(k, n)$

A competitive equilibrium is a sequence of allocations $\{c_t^i, l_t^i, k_{t+1}^i, s_{t+1}^i, e_t^i\}_{i\in I, t\in\mathbb{N}}$, output $\{y_t^{c,j}, k_t^{c,j}, n_t^{c,j}\}_{j\in J^c, t\in\mathbb{N}}$, $\{y_t^{x,j}, k_t^{x,j}, n_t^{x,j}\}_{j\in J^x, t\in\mathbb{N}}$, $\{y_t^{e,j}, k_t^{e,j}, n_t^{e,j}\}_{j\in J^e, t\in\mathbb{N}}$, and prices $\{p_{c,t}, p_{x,t}, p_{e,t}, r_t, w_t\}$ such that:

1. Households maximize utility taking prices as given (dropping individual identifier i):

$$\max_{\{c_t, l_t, k_{t+1}, s_{t+1}, e_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \tag{5}$$

$$s.t. \quad p_{c,t}c_t + p_{x,t}x_t + p_{e,t}e_t \leq r_t k_t + w_t(e_t + s_t - l_t)$$

$$s_{t+1} = (1-\delta_e)s_t + e_t \tag{6}$$

$$x_t = k_t + 1 - (1-\delta)k_t$$

$$k_0, s_0 \text{ given} \tag{7}$$

2. Firms maximize profits taking prices as given (dropping firm identifier j):

$$\begin{aligned} \text{Consumption Good Producers: } & \max_{n_t^c, k_t^c} p_{c,t} F(k_t^c, n_t^c) - w_t n_t^c - r_t k_t^c \\ \text{Investment Good Producers: } & \max_{n_t^x, k_t^x} p_{x,t} F(k_t^x, n_t^x) - w_t n_t^x - r_t k_t^x \\ \text{Education Producers: } & \max_{n_t^e, k_t^e} p_{e,t} F^e(k_t^e, n_t^e) - w_t n_t^e - r_t k_t^e \end{aligned} \tag{8}$$

3. Markets Clear:

$$\begin{aligned} \sum_{i \in I} c_t &= \sum_{j \in J^c} Y_t^c \\ \sum_{i \in I} x_t &= \sum_{j \in J^x} Y_t^x \\ \sum_{i \in I} e_t &= \sum_{j \in J^e} Y_t^e \\ \sum_{i \in I} k_t &= k_t^c + k_t^k + k_t^h \\ \sum_{i \in I} (e_t + s_t - l_t) &= n_t^c + n_t^k + n_t^h \end{aligned} \tag{9}$$

Problem 7

In this problem we extend the model to capture the interaction of environmental factors with the economy. The assignment asks to consider emissions that harm labor productivity. One straigthforward way to model this feature is to consider that emissions reduce the time endowment of workers.

To capture this aspect, we modify the budget constraint of agents to include emissions. We assume that the total amount of emissions depends on aggregate production. It could be a function of all past aggregate production $E_t = f(\{Y_s\}_{s=0}^t)$ or have zero persistence and depend on the current production only $E_t = f(Y_t)$. In any case, f is increasing in its arguments, indicating that higher levels of production bring about more pollution. Then it is necessary to make some assumptions on how much total emissions affect each individual i: $E_t^i = g_i(E_t, I)$ with $\frac{\partial E_t^i}{\partial E_t} > 0 \; \forall i \in I, \; \forall t$ where (with some abuse of notation) I denotes the set of agents and their features.

A competitive equilibrium is a sequence of allocations $\{c_t^i, l_t^i, x_t^i, k_{t+1}^i, E_t^i\}_{i,t}$, output $\{y_t, k_t, n_t\}$, and prices $\{p_t, r_t, w_t\}$ such that:

1. Households maximize utility taking prices as given (dropping individual identifier i):

$$\max_{\{c_t^i, l_t^i, k_{t+1}^i, n_t^i\}} \sum_{t=0}^{\infty} \beta^t u(c_t^i, l_t^i) \; s.t.$$
$$\sum_{t=0}^{\infty} p_t(c_t^i + x_t^i) \leq \sum_{t=0}^{\infty} r_t k_t^i + w_t n_t^i$$
$$x_t^i = k_{t+1}^i - (1-\delta)k_t^i$$
$$l_t^i + n_t^i \leq e_t^i - E_t^i$$
$$k_0^i \text{ given}$$

Note that households do not have the pollution as a choice variable, because each individual is infinitesimal compared to the whole population, and therefore cannot affect the total output-and pollution-by its consumption and leisure choices.

2. Firms maximize profits taking prices as given (dropping firm identifier j and assuming that consumption and investment goods are one and the same):

$$\max_{n_t, k_t} p_t y_t - w_t n_t - r_t k_t \; s.t.$$
$$y_t = F(k_t, n_t)$$

3. Markets Clear:

$$\sum_{i \in I} c_t^i + x_t^i = \sum_{j \in J} y_t^j$$
$$\sum_{i \in I} k_t^i = \sum_{j \in J} k_t^j$$
$$\sum_{i \in I} n_t^i = \sum_{j \in J} n_t^j$$

4. Emissions:

$$E_t = f(Y_t) = f(\sum_{j \in J} y_t^j)$$
$$E_t^i = g_i(E_t, I) \; \forall i$$

Part (b)

A Pareto optimal allocation is a sequence of allocations $\{c_t^i, l_t^i, k_{t+1}^i, E_t^i\}_{i\in I, t\in\mathbb{N}}$, $\{y_t^j, k_t^j, n_t^j\}_{j\in J, t\in\mathbb{N}}$ that are feasible and are such that no different feasible allocation makes all agents weakly better off and at least one agent strictly better off.

With externalities the C.E. outcome is not Pareto efficient. Firms do not bear the cost of the pollution, but society does, resulting in excess production. To see this, note that a social planner would solve the following problem instead

$$\max_{\{C_t,L_t,K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t, l_t, E_t) \ s.t.$$

$$C_t^i + K_{t+1} - (1-\delta)K_t \leq F(K_t, n_t)$$

$$\sum_i (e_t^i - E_t^i - l_t^i) = n_t$$

$$E_t = f(F(K_t, n_t))$$

$$E_t^i = g_i(E_t, I)) \ \forall i \tag{10}$$

$$K_0 \text{ given} \tag{11}$$

Which results in a lower level of production.

Part (c)

In order to prove that the first welfare theorem does not hold, it is sufficient to find one circumstance in which competitive equilibirum is not Pareto optimal. To do so, consider the following competitive equilibrium in a static economy with one agent, one firm, and one consumption good:

Household

$$\max_{\{c,\ell\}} \ln c + e - E - \ell$$

$$s.t. \quad c \leq rk + w\ell$$

$$\ell \leq e - E$$

$$k \text{ given}$$

Firm

$$\max_{\{K,L\}} K^\alpha L^{1-\alpha} - rK - wL$$

Market clearing

$$\ell = L$$

$$k = K$$

$$c = Y = K^{\alpha} L^{1-\alpha}$$

Emissions E are a function of production $E = f(Y)$. To simplify, assume $E = Y$. e is the time endowment.

At equilibrium:
$FOC_{HH} : c^{-1} = \lambda \wedge w^{-1} = \lambda \implies c = w$
$FOC_{Firm} : \alpha \left(\frac{L}{K}\right)^{1-\alpha} = r \wedge (1-\alpha)\left(\frac{K}{L}\right)^{\alpha} = w$

Joining firm's FOC with respect to labor, household FOC, and consumption good market clearing:
$(1-\alpha)\left(\frac{K}{L}\right)^{\alpha} = c = K^{\alpha} L^{1-\alpha} \implies L^*_{CE} = \ell^*_{CE} = (1-\alpha)$
Which gives $c^*_{CE} = K^{\alpha}(1-\alpha)^{1-\alpha}$ and $E^*_{CE} = K^{\alpha}(1-\alpha)^{1-\alpha}$ and hence

$$U(c^*_{CE}, l^*_{CE}) = \ln\left(K^{\alpha}(1-\alpha)^{1-\alpha}\right) + e - K^{\alpha}(1-\alpha)^{1-\alpha} - (1-\alpha)$$

$$E = Y = K^{\alpha} L^{1-\alpha}$$

Now compare this value with the value of the planner's problem for the same economy, defined as follows:

$$\max_{\{c,\ell,L,E\}} \ \ln c + e - E - \ell$$

$$s.t. \quad c \leq K^{\alpha} L^{1-\alpha}$$

$$\ell \leq e - E$$

$$L \leq \ell$$

$$E = K^{\alpha} L^{1-\alpha}$$

Which corresponds to the unconstrained optimization:

$$\max_{\{c,\ell,L,E\}} \ln c + e - E - \ell + \lambda(K^\alpha L^{1-\alpha} - c) + \mu(e - E - \ell) + \eta(\ell - L) + q(K^\alpha L^{1-\alpha} - E)$$

At equilibrium:

$c^{-1} = \lambda$

$1 - \mu - \eta = 0$

$\lambda(1-\alpha)\left(\frac{K}{L}\right)^\alpha - \eta + q(1-\alpha)\left(\frac{K}{L}\right)^\alpha = 0$

$-\mu - q - 1 = 0$

And $\mu = 0$ if it is assumed that the time endowment is large enough, since it will not be optimal to use up all the available time to work. Hence:

$0 = -q - 1 \implies q = 1 \implies$

$$\begin{aligned} c^{-1}(1-\alpha)\left(\frac{K}{L}\right)^\alpha - 1 - (1-\alpha)\left(\frac{K}{L}\right)^\alpha &= 0 \\ K^{-\alpha}L^{\alpha-1}(1-\alpha)\left(\frac{K}{L}\right)^\alpha - 1 - (1-\alpha)\left(\frac{K}{L}\right)^\alpha &= 0 \\ K^{-\alpha}L^{\alpha-1} - (1-\alpha)^{-1}\left(\frac{K}{L}\right)^{-\alpha} - 1 &= 0 \\ K^{-\alpha}L^\alpha\left(L^{-1} - \frac{1}{1-\alpha}\right) &= 1 \\ K = L\left(L^{-1} - \frac{1}{1-\alpha}\right)^{\frac{1}{\alpha}} \end{aligned}$$

Since $K > 0$ this imposes that $L^{-1} = \ell^{-1} > \frac{1}{1-\alpha} \implies \ell^*_{Planner} < 1 - \alpha = \ell^*_{CE}$.

The planner's problem solution being incompatible with the CE shows that CE is not PO.

In fact, we can see that the derivative of $U(c, \ell)$ evaluated at ℓ^*_{CE}, c^*_{CE} is negative:

$$\frac{\partial}{\partial 1-\alpha} \ln\left(K^\alpha(1-\alpha)^{1-\alpha}\right) + e - K^\alpha(1-\alpha)^{1-\alpha} - (1-\alpha) = \tag{12}$$

$$\frac{1}{K^\alpha(1-\alpha)^{1-\alpha}} K^\alpha(1-\alpha)(1-\alpha)^{-\alpha} - K^\alpha(1-\alpha)(1-\alpha)^{-\alpha} - 1 = \tag{13}$$

$$1 - K^\alpha(1-\alpha)(1-\alpha)^{-\alpha} - 1 < 0 \tag{14}$$

And feasibility indeed allows the reduction of work from $\ell = 1 - \alpha$

Part (d)

Any policy that forces firms to decrease production to efficient levels will achieve Pareto optimality. Firms could compensate workers for the marginal effect on utility of emissions will reduce production to efficient levels. The textbook solution is a Pigouvian tax paid by the firms to the individuals that is equal to the marginal social cost of emissions.