Problem 1

Part (a)

We can write this problem as

$$\sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{\left(\frac{Ak_t - k_{t+1}}{h_t}\right)^{1-\sigma}}{1-\sigma} \right) \; s.t.$$

$$k_{t+1} \in (0, Ak_t), \; \forall t \geq 0 \tag{1}$$

$$h_{t+1} = (Ak_t - k_{t+1})^{\delta} h_t^{1-\delta}, \; \forall t \geq 0$$

$$k_0, h_0 \text{ given}$$

Part (b)

Writing the problem in recursive form:

$$V(k,h) = \max_{k'} \left[\left(\frac{\left(\frac{Ak-k'}{h_t}\right)^{1-\sigma}}{1-\sigma} \right) + \beta V(k', (Ak-k')^{\delta} h^{1-\delta}) \right] \; s.t.$$

$$\Gamma(k) = [0, Ak] \tag{2}$$

Note that we cannot use the standard tools from Stokey Lucas to prove that the solution to this functional equation exists, because here the return function (Ak) is unbounded.

Part (c)

The problem should read that the value function associated with the problem is homogeneous of degree $1-\sigma$. Suppose it was not homogeneous of degree $1-\sigma$. We have two cases to consider, where an increase of endowments by ε and holding technology fixed increases utility by less than $\varepsilon^{1-\sigma}$ or greater than $\varepsilon^{1-\sigma}$. If it increases utility by less than $\varepsilon^{1-\sigma}$, this implies the agent is increasing consumption by less than ε. But with endowments increased by ε and technology fixed, the agent could increase consumption to ε and increase utility. Similarly if it increases utility by more than $\varepsilon^{1-\sigma}$, this implies that when the endowment was ε smaller, the agent was not optimizing, implying that function was not their true value function as it didn't maximize utility.

With this fact in hand, we can normalize one of the state variables to one,

$$V(k,h) = \left(\frac{1}{h}\right)^{1-\sigma} v\left(\frac{k}{h}, 1\right)$$

We define our new single state varialbe as $z = \frac{k}{h}$ and rewrite our Bellman equation as

$$V(z) = \frac{(Az - z')^{1-\sigma}}{1-\sigma} + \beta V(z)$$

Part (d)

Now we need to determine the domain of the new state variable $\hat{h}$. Remember that since $k = 1$ in $V(1, \hat{h})$ we have:

$$\hat{h}' = \frac{h'}{k'} = \frac{\alpha(A - k')}{k'} \rightarrow \hat{h}' = \frac{\alpha A}{k'} - \alpha$$

Since $k' \in (\alpha, A - \hat{h})$, we have that $\hat{h}' \in (\frac{\alpha \hat{h}}{A - \hat{h}}, A - \alpha)$. Next, note that from the policy and habit functions, (**??**) and (**??**), which I derived in part (c), we can derive $\hat{h}'$, by setting $k = 1$:

$$\hat{h}' = \frac{\alpha(A - \alpha)(1 - \beta) + \alpha\beta\hat{h}}{A\beta + (1 - \beta)\alpha - \beta\hat{h}}$$

First, observe that

$$\frac{d\hat{h}'}{d\hat{h}} = \frac{\alpha\beta\left(A\beta + (1-\beta)\alpha - \beta\hat{h}\right) + \beta\left(\alpha(A-\alpha)(1-\beta) + \alpha\beta\hat{h}\right)}{\left(A\beta + (1-\beta)\alpha - \beta\hat{h}\right)^2} = \frac{\alpha\beta\left(A\beta + \alpha(1-\beta) + (A-\alpha)(1-\beta)\right)}{\left(A\beta + (1-\beta)\alpha - \beta\hat{h}\right)^2} > 0.$$

Showing that the policy function $\hat{h}'(\hat{h})$ is increasing in $\hat{h}$. Next, note that we can write the equation

$$\hat{h}' = \hat{h} \rightarrow \frac{\alpha(A - \alpha)(1 - \beta) + \alpha\beta\hat{h}}{A\beta + (1 - \beta)\alpha - \beta\hat{h}} = \hat{h}$$

As

$$\left(\hat{h} - (A - \alpha)\right)\left(\hat{h} - \frac{\alpha(1 - \beta)}{\beta}\right) = 0$$

Hence, for $0 < \hat{h} < \frac{\alpha(1-\beta)}{\beta}$, we have $\hat{h} < \hat{h}'$, and since I have already shown that $\hat{h}'(\hat{h})$ is increasing in $\hat{h}$, we get that the policy sequence is increasing in $\hat{h} \in (0, \frac{\alpha(1-\beta)}{\beta})$.
On the other hand if $\frac{\alpha(1-\beta)}{\beta} < \hat{h} < A - \alpha$, we get that $\hat{h}' < \hat{h}$, and again since $\hat{h}'(\hat{h})$ is increasing in $\hat{h}$, policy sequence is decreasing in $\hat{h} \in (\frac{\alpha(1-\beta)}{\beta}, A - \alpha)$.. Therefore, for any $\hat{h} \in (0, A - \alpha)$ the additional assumptions needed for theorems 4.3 and 4.5 of SLP are satisfied. Hence, we can be sure that this value function is the unique function satisfying the functional equation that satisfies those additional assumptions.

Part (e)

Take the policy function and habit stock

$$k' = (A\beta + (1 - \beta)\alpha)k - \beta h$$

$$h' = \alpha(A - \alpha)(1 - \beta)k - \alpha\beta h$$

Divide k' and h' by k and h respectively

$$\frac{k'}{k} = (A\beta + (1-\beta)\alpha) - \beta\frac{h}{k}$$
$$\frac{h'}{h} = \alpha(A-\alpha)(1-\beta)\frac{k}{h} - \alpha\beta$$

I have already shown in part (d) the steady state value for $\frac{h}{k} = \frac{\alpha(1-\beta)}{\beta}$, and I have reasoned why $\hat{h}$ converges to that. Therefore I only need to characterize the BGP where $\hat{h}$ reaches its steady state value and capital, k, and habit stock, h, grow at a constant rate

$$\frac{k'}{k} = (A\beta + (1-\beta)\alpha) - \beta\frac{h}{k} = A\beta$$
$$\frac{h'}{h} = \frac{c'}{c} = \alpha(A-\alpha)(1-\beta)\frac{k}{h} - \alpha\beta = A\beta$$

Hence, on BGP the growth rate of capital and consumption is equal to $A\beta$.

Part (f)

Starting from an initial value $0 < \hat{h} < \frac{\alpha(1-\beta)}{\beta}$, I proved that we have $\hat{h} < \hat{h}'$. By $\frac{dh'}{d\hat{h}} > 0$, we have that the policy sequence increases towards the steady state point.
Similarly starting from an initial value $\frac{\alpha(1-\beta)}{\beta} < \hat{h} < A-\alpha$, we have $\hat{h}' < \hat{h}$, which means the policy sequence will decrease towards its steady state. This means if the economy has a low habit stock to capital ratio, agents consume more and buid up a higher habit stock. On the other hand, if the habit stock is relatively large with respect to the capital, the economy will move towards investing more and lowering the ratio of habit stock relative to capital.

Problem 2

Part (a)

$$\max_{\{c_t\}_t} W(\mathbf{c}) \; s.t.$$
$$c_t + k_{t+1} - (1-\delta)k_t \leq f(k_t)$$
$$k_0 \text{ given}$$

Denote by $g(k, k') := f(k) + (1-\delta)k - k'$. The associated Bellman equation is

$$V(k) = \max_{k'} G(g(k,k'), V(k')) \; s.t.$$
$$k' \in [(1-\delta)k, Ak + (1-\delta)k]$$

Part (b)

- G_1: continuous and bounded
- G_2: concave (not necessary for this exercise)
- G_3: $G(0,0) = 0$
- G_4: $(x,z) \le (x,z)$ and $(x',z') \ne (x,z)$ implies $G(x,z) < G(x',z')$
  and for some $0 \le \beta < 1$,
- G_5: $|G(x,z) - g(x,z')| \le \beta |z - z'|$ for all $x \in R_+^m$ and all $z, z' \in R_+$.

Now I will show that given this set of assumptions, we can prove that there is exactly one continuous bounded function V satisfying

$$V(k) = \max_{k'} G(g(k,k'), V(k'))\ s.t.$$
$$k' \in [(1-\delta)k, Ak + (1-\delta)k]$$

Let C be the Banach space of continuous bounded functions f with norm

$$||f|| = sup_k |f(k)|$$

Let T be the operator on C defined by $(Tf)(k) = \max_{k'} G(g(k,k'), f(k'))$. The set $[(1-\delta)k, Ak + (1-\delta)k]$ is compact and the function to be maximized in applying T is continuous, so Tf is well defined for each $f \in C$. Tf is continuous and evidently bounded, so that $T : C \to C$. Next, we show that T is a contraction mapping, by showing that it satisfies the Blackwell's sufficient conditions

$$f \le f' \to Tf \le Tf', \qquad \text{for all } f, f' \in C$$

and for some $\beta \in [0,1)$

$$T(f + a) \le Tf + \beta a, \qquad \text{for all } f \in C \text{ and all } a > 0$$

Note that the monotonicity condition follows from G_4, i.e. $(x,z) \le (x',z')$ and $(x',z') \ne (x,z)$ implies $G(x,z) < G(x',z')$. Since we have $f(k) + (1-\delta)k - k' \le f'(k) + (1-\delta)k - k'$ and $f(k') \le f'(k')$, we get that $G(g(k,k'), f(k')) \le G(g'(k,k'), f'(k'))$.
Next since $[(1-\delta)k, f(k) + (1-\delta)k] \subset [(1-\delta)k, f'(k) + (1-\delta)k]$, we get that $Tf' \ge Tf$.
The second property, discounting, follows directly from G_4 and G_5, i.e. $|G(x,z) - G(x,z')| \le \beta|z - z'|$ for all $x \in R_+^m$ and all $z, z' \in R_+$.

Let $k'^* = armax_{k'} G(g(k,k'), f(k') + a)$. By G_5 we have

$$|G(g(k,k'^*), f(k'^*) + a) - G(g(k,k'^*), f(k'^*))| \leq \beta a$$

Note that by G_4, $G(g(k,k'^*), f(k'^*)) \leq G(g(k,k'^*), f(k'^*) + a)$. Therefore we get that

$$G(g(k,k'^*), f(k'^*) + a) \leq G(g(k,k'^*), f(k'^*)) + \beta a \leq (Tf)(k) + \beta a.$$

Hence, by contraction mapping theorem, T has a unique fixed point, which is the solution to our functional equation. Epstein Zin preferences are an example of recursive preferences.

Part (c)

Taking the first order condition

$$\frac{dG}{dx}\frac{dg}{dk'} + \frac{dG}{dz}.V'(k') = 0$$

Envelope condition

$$V'(k) = \frac{dG}{dx}.\frac{dg}{dk} = \frac{dG}{dx}.(f'(k) + 1 - \delta)$$

In steady state $k' = k$

$$\frac{dG}{dx}\frac{dg}{dk'} + \frac{dG}{dz}.V'(k') = \frac{dG}{dx}\frac{dg}{dk'} + \frac{dG}{dz}.V'(k) = \frac{dG}{dx}\frac{dg}{dk'} + \frac{dG}{dz}.\frac{dG}{dx}.(f'(k) + 1 - \delta) = 0 \rightarrow$$

$$\frac{dg}{dk'} + \frac{dG}{dz}.(f'(k) + 1 - \delta) = 0 \rightarrow \frac{dG}{dz}.(f'(k) + 1 - \delta) = 1.$$

Problem 3

This problem is studied in Greenwood, Hercowitz and Per Krusell (1997) - see the paper if you want to learn more about it.

Part (a)

For this problem I ignore profits (which will be zero at the CE). I assume that households have a fixed endowment of leisure $\bar{e} = 1$ and that leisure is unvalued. A competitive equilibrium is prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$ and allocations $\{c_t^i, x_t^i, k_t^i, y_t^f, n_t^f, k_t^f\}_{t=0}^{\infty}$ such that

1. Households solve the problem

$$\max_{\{c_t^i, x_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \ s.t.$$

$$p_t(c_t^i + x_t^i) \leq r_t k_t^i + w_t \frac{1}{I}$$

$$k_{t+1} = (1-\delta)k_t + q_t x_t$$

$$q_{t+1} = (1+g_q)q_t$$

$$k_{t+1} \geq 0 \ \forall t$$

$$k_0 \text{ given}$$

2. The representative firm maximizes profits:

$$\max_{y_t^f, k_t^f} p_t y_t^f - r_t k_t^f - w_t n_t^f \ s.t.$$

$$y_t^f = z(k_t^f)^{\alpha}(n_t^f)^{1-\alpha}$$

3. Markets clear:

$$\sum_i \left(c_t^i + x_t^i\right) = y_t^f$$

$$1 = n_t^f$$

$$\sum_i k_t^i = k_t^f \qquad \forall \ i$$

Part (b)

First note that on any balanced growth path, both sides of the investment equation

$$k_{t+1} - (1-\delta)k_t = x_t q_t$$

have to grow at the same rate. If $(1+g_k)$ is the growth rate of capital, and $(1+g_x)$ the growth rate of investment, then we have

$$1 + g_k = (1+g_x)(1+g_q)$$

Additionally, from the production function we must have that at any BGP,

$$(1+g_y) = (1+g_k)^{\alpha}$$

Since $\alpha < 1$, it is clearly the case that $g_k > g_y$. Supposing that $g = g_y = g_x$ (i.e. investment and output [and consumption] all grow at the same rate), we have

$$1 + g_k = (1 + g)(1 + g_q)$$
$$1 + g = (1 + g_k)^{\alpha}$$

and hence

$$1 + g = (1 + g_q)^{\frac{\alpha}{1-\alpha}}$$
$$1 + g_k = (1 + g_q)^{\frac{1}{1-\alpha}} \tag{1}$$

Part (c)

To write the planner's problem recursively, we need to first put the model in terms of a steady state, which means dividing out by the growth rates. Denote the following:

$$\hat{c}_t = c_t/(1 + g)^t$$
$$\hat{x}_t = x_t/(1 + g)^t$$
$$\hat{u}_t = y_t/(1 + g)^t$$
$$\hat{k}_t = k_t/(1 + g_k)^t$$
$$\hat{q}_t = q_t/(1 + g_q)^t$$

Then

$$\begin{aligned}
\hat{x}_t &= \frac{k_{t+1} - (1 - \delta)k_t}{(1 + g_q)^{\frac{t\alpha}{1-\alpha}} q_t} \\
&= \frac{(1 + g_q)^{\frac{t+1}{1-\alpha}} \hat{k}_{t+1} - (1 + g_q)^{\frac{t}{1-\alpha}} (1 - \delta)\hat{k}_t}{(1 + g_q)^{\frac{t\alpha}{1-\alpha}} (1 + g_q)^t \hat{q}_t} \\
&= \frac{(1 + g_q)^{\frac{1}{1-\alpha}} \hat{k}_{t+1} - (1 - \delta)\hat{k}_t}{\hat{q}_t} \\
&= \frac{(1 + g_q)^{\frac{1}{1-\alpha}} \hat{k}_{t+1} - (1 - \delta)\hat{k}_t}{\hat{q}}
\end{aligned}$$

where the last equality follows from the fact that $\hat{q}_{t+1}/\hat{q}_t = 1$. And in yet another example of the wonders of Cobb-Douglas:

$$\begin{aligned}\hat{y}_t &= \frac{zk_t^\alpha}{(1+g_q)^{\frac{t\alpha}{1-\alpha}}} \\ &= \frac{z(1+g_q)^{\frac{t\alpha}{1-\alpha}}\hat{k}_t^\alpha}{(1+g_q)^{\frac{t\alpha}{1-\alpha}}} \\ &= z\hat{k}_t^\alpha\end{aligned}$$

We therefore have the planner's problem

$$\max_{\{\hat{c}_t,\hat{k}_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(\hat{c}_t) \ s.t.$$

$$\hat{c}_t + \frac{(1+g_q)^{\frac{1}{1-\alpha}}}{\hat{q}}k_{t+1} = \frac{1-\delta}{\hat{q}}\hat{k}_t + z\hat{k}_t^\alpha$$

Substituting the feasibility constraint into the utility function and taking the FOC to get the Euler equation:

$$\frac{(1+g_q)^{\frac{1}{1-\alpha}}}{\hat{q}}u_{c,t} = \beta u_{c,t+1}\left(z\alpha\hat{k}_{t+1}^{\alpha-1} + \frac{1-\delta}{\hat{q}}\right)$$

and rewriting,

$$\frac{u_{c,t}}{u_{c,t+1}} = \frac{\beta}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(\hat{q}z\alpha\hat{k}_{t+1}^{\alpha-1} + 1 - \delta\right) \tag{2}$$

If the economy starts with a low level of capital, the RHS of (2) is greater than 1 - in which case the left-hand side is also greater than 1, which implies that consumption is growing (and the economy accumulating capital) given the standard assumption that u is strictly concave. On the other hand, as capital goes towards infinity, we have

$$\lim_{\hat{k}\to\infty} \frac{\beta}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(\hat{q}z\alpha\hat{k}_{t+1}^{\alpha-1} + 1 - \delta\right) = \frac{\beta(1-\delta)}{(1+g_q)^{\frac{1}{1-\alpha}}}$$

which is clearly less than 1, indicating that investment is negative and consumption is declining over time. Hence the economy will eventually converge to a steady state where the LHS of (2) is equal to 1, and at this level we will have

$$\hat{k} = \left(\frac{\beta^{-1}(1+g_q)^{\frac{1}{1-\alpha}} - (1-\delta)}{\hat{q}z\alpha}\right)^{\frac{1}{\alpha-1}} \tag{3}$$

Part (d)

The rental price of capital will be the marginal return of capital:

$$\begin{aligned} r_t &= z\alpha k_t^{\alpha-1} \\ &= z\alpha \left(\frac{1}{k_t}\right)^{1-\alpha} \end{aligned}$$

showing that the ratio of labor to capital is decreasing, and hence the rental rate of capital is decreasing. At the BGP we can write the rental rate explicitly as

$$\begin{aligned} r_t &= z\alpha \left(\frac{1}{k_0(1+g_q)^{\frac{t}{1-\alpha}}}\right)^{1-\alpha} \\ &= z\alpha k_0^{\alpha-1}(1+g_q)^{-t} \end{aligned} \tag{4}$$

Now at any equilibrium, we must have the return to saving equal to the return to investing. From (2) above, we have

$$\begin{aligned} \frac{u_{c,t}}{\beta u_{c,t+1}} &= i_t \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}} \left(\hat{q} z\alpha \hat{k}_{t+1}^{\alpha-1} + 1 - \delta\right) \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}} \left(\frac{q_t}{(1+g_q)^t} z\alpha \left(\frac{k_t}{(1+g_q)^{\frac{t}{1-\alpha}}}\right)^{\alpha-1} + 1 - \delta\right) \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}} \left(q_0(1+g_q)^t z\alpha k_0^{\alpha-1}(1+g_q)^{-t} + 1 - \delta\right) \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}} \left(q_0 z\alpha k_0^{\alpha-1} + 1 - \delta\right) \end{aligned} \tag{5}$$

showing that the interest rate is constant at the BGP. Basically: the return to capital *per unit* is falling, but because units of capital become cheaper over time relative to consumption, the return to investment (and hence the return to saving) is constant.

Part (e)

Repeating the steps in part (b), we arrive at the system of equations:

$$\begin{aligned} (1+g_e) &= (1+g)(1+g_q) \\ (1+g) &= (1+g_z)(1+g_e)^{\alpha_e}(1+g)^{\alpha_s} \end{aligned}$$

with solutions

$$1+g_e=(1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}$$
$$1+g=(1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}$$

Hence equipment grows at a faster rate than structures, and so the ratio of equipment to structures is growing over time. Now we wish to examine the rental rate and investment rate. Using the same notation as before, we have

$$\begin{aligned}
\hat{x}_t &= \frac{\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t+1}\hat{k}^e_{t+1}-(1-\delta_e)\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t}\hat{k}^e_t}{\left((1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t}(1+g_q)^t\hat{q}_t} \\
&= \frac{\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t+1}\hat{k}^e_{t+1}-(1-\delta_e)\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t}\hat{k}^e_t}{\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t}\hat{q}_t} \\
&= \frac{(1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\hat{k}_{t+1}-(1-\delta)\hat{k}_t}{\hat{q}} \\
\hat{i}_t &= (1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\hat{k}^s_{t+1}+(1-\delta_s)\hat{k}^s_t
\end{aligned}$$

And if we follow the same steps as in part (d), we obtain the rental rate

$$\begin{aligned}
r_t &= z_0(1+g_z)^t\alpha\left(\frac{1}{k^e_0(1+g_q)^{\frac{t-t\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{t}{1-\alpha_s-\alpha_e}}}\right)^{1-\alpha_e}\left(k^s_0(1+g_q)^{\frac{t\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{t}{1-\alpha_s-\alpha_e}}\right)^{\alpha_s} \\
&= z_0\alpha_e(k^e_0)^{\alpha_e-1}(k^s_0)^{\alpha_s}(1+g_q)^{-t(1-\alpha_s-\alpha_e)} \qquad (6)
\end{aligned}$$

which is again decreasing over time. Likewise the investment rate will be

$$\begin{aligned}
i_t &= \frac{\beta}{(1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}}\left(\hat{q}\hat{z}_t\alpha_e(\hat{k}^e_t)^{\alpha_e-1}(\hat{k}^s_t)^{\alpha_s}+1-\delta\right) \\
&= \frac{\beta}{(1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}}\left(q_0\alpha_e(k^e_0)^{\alpha_e-1}(k^s_0)^{\alpha_s}+1-\delta\right) \qquad (7)
\end{aligned}$$

which remains constant as before.

Problem 4

Part (a)

Let $p_{1,t}$ and $p_{2,t}$ denote the prices of good 1 and good 2, respectively. Let $x_{1,t}$ denote the investment of good 1. A competitive equilibrium is a sequence of allocations $\{c_{1,t},c_{2,t},k_{t+1}\}$, $\{y_{1,t},k_{1,t},n_{1,t}\}$,

$\{y_{2,t}, k_{2,t}, n_{2,t}\}$ and prices $\{p_{1,t}, p_{2,t}, r_t, w_t\}$ such that:

1. Households maximize utility taking prices as given:

$$\max_{\{c_{1,t}, c_{2,t}, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [a \log c_{1,t} + (1-a) \log(c_{2,t} + \bar{c})] \ s.t.$$

$$\sum_{t=0}^{\infty} p_{1,t}(c_{1,t} + x_{1,t}) + p_{2,t} c_{2,t} \leq \sum_{t=0}^{\infty} r_t k_t + w_t n_t$$

$$k_{t+1} = (1-\delta) k_t + x_{1,t}, \ \forall t \geq 0$$

$$k_{t+1}, c_t \geq 0; \ \forall t \geq 0$$

$$k_0 \text{ given}$$

2. Firms maximize profits taking prices as given:

$$\text{Good 1 Sector: } \max_{n_{1,t}, k_{1,t}} p_{1,t} k_{1,t}^{\alpha} (A_{1,t} n_{1,t})^{1-\alpha} - w_t n_{1,t} - r_t k_{1,t}$$

$$\text{Good 2 Sector: } \max_{n_{2,t}, k_{2,t}} p_{2,t} k_{2,t}^{\alpha} (A_{2,t} n_{2,t})^{1-\alpha} - w_t n_{2,t} - r_t k_{2,t}$$

3. Markets Clear:

$$\sum_i [c_{1,t} + x_{1,t}] = k_{1,t}^{\alpha} (A_{1,t} n_{1,t})^{1-\alpha}$$

$$\sum_i c_{2,t} = k_{2,t}^{\alpha} (A_{2,t} n_{2,t})^{1-\alpha}$$

$$\sum_i k_t = k_{1,t} + k_{2,t}$$

$$\sum_i n_t = n_{1,t} + n_{2,t} = 1$$

Part (b)

The solution to the *Competitive Equilibrium* defined above coincides with the *Pareto Optimal* solution. The problem becomes:

$$\max_{\{c_{1,t},c_{2,t},k_{t+1},n_{1,t},n_{2,t}\}} \sum_{t=0}^{\infty} \beta^t [a \log c_{1,t} + (1-a)\log(c_{2,t}+\bar{c})] \ s.t.$$

$$c_{2,t} = k_{2,t}^{\alpha}(A_{2,t}n_{2,t})^{1-\alpha} \qquad (\lambda_{1,t})$$

$$k_{t+1} = k_{1,t}^{\alpha}(A_{1,t}n_{1,t})^{1-\alpha} + (1-\delta)k_t - c_{1,t} \qquad (\lambda_{2,t})$$

$$n_{1,t} + n_{2,t} = 1$$

$$k_{1,t} + k_{2,t} = k_t$$

$$k_0 \text{ given}$$

Part (c)

We can use the Pareto set-up to write the problem recursively.

$$V(k) = \max_{k'} \Bigg[\operatorname*{argmax}_{n_1,n_2,k_1,k_2} \Bigg\{ a \log \Big((k-k_2)^{\alpha}(A_1 n_1)^{1-\alpha} + (1-\delta)(k) - k' \Big) + (1-a)\log\Big((k-k_1)^{\alpha}(A_2 n_2)^{1-\alpha} + \bar{c} \Big) \Bigg\} + V(k') \Bigg]$$

Here there is the intertemporal optimization problem which is characterized by choice over total capital next period k', and within each period the representative agent must choose the optimal allocation of labor and capital between sectors, taking overall level of capital k as fixed. Note Zahra's solution from last year solving for the indirect utility function with a nice closed form solution doesn't apply here with Stone-Geary preferences.

SP First Order Conditions

$$[c_{1,t}]: \ \frac{\beta^t a}{c_{1,t}} - \lambda_{1,t} = 0 \tag{1}$$

$$[c_{2,t}]: \ \frac{\beta^t a}{c_{2,t} + \bar{c}} - \lambda_{2,t} = 0 \tag{2}$$

$$[k_{1,t+1}]: \ -\lambda_{1,t} + \lambda_{1,t+1}(1-\delta) + \lambda_{1,t+1}\alpha k_{1,t+1}^{\alpha-1}(A_{1,t+1}n_{1,t+1})^{1-\alpha}) = 0 \tag{3}$$

$$[k_{2,t+1}]: \ -\lambda_{1,t} + \lambda_{1,t+1}(1-\delta) + \lambda_{2,t+1}(\alpha k_{2,t+1}^{\alpha-1}(A_{2,t+1}n_{2,t+1})^{1-\alpha}) = 0 \tag{4}$$

From (1) and (2), we observe:

$$\frac{c_{1,t}}{c_{2,t}+\bar{c}} = \frac{a}{1-a}\frac{\lambda_{2,t}}{\lambda_{1,t}}$$
$$\frac{c_{1,t+1}}{c_{1,t}} = \beta\frac{\lambda_{1,t}}{\lambda_{1,t+1}}$$
$$\frac{c_{2,t+1}+\bar{c}}{c_{2,t}+\bar{c}} = \beta\frac{\lambda_{2,t}}{\lambda_{2,t+1}}$$

From (3) and (4), we observe:

$$\frac{\lambda_{1,t+1}}{\lambda_{2,t+1}} = \left(\frac{A_{2,t+1}}{A_{1,t+1}}\right)^{1-\alpha}$$
$$\frac{\lambda_{1,t}}{\lambda_{1,t+1}} = 1-\delta+\alpha k_{1,t+1}^{\alpha-1}(A_{1,t+1}n_{1,t+1})^{1-\alpha}$$

Combining the above we see that

$$\frac{c_{1,t+1}}{c_{1,t}} = \beta(1-\delta+\alpha k_{1,t+1}^{\alpha-1}(A_{1,t+1}n_{1,t+1})^{1-\alpha})$$

Iterating this equation back one period and combining with previous equation, we have:

$$\frac{c_{2,t}+\bar{c}}{c_{1,t}} = \frac{1-a}{a}\left(\frac{A_{2,t}}{A_{1,t}}\right)^{1-\alpha}$$

On a balanced growth path, the endogenous variables are changing at a constant rate. Therefore

$$\frac{c_{1,t+1}}{c_{1,t}} = \beta(1-\delta+\alpha k_{1,t+1}^{\alpha-1}(A_{1,t+1}n_{1,t+1})^{1-\alpha}) = 1+g_{c_1}$$

Iterating back and dividing we must have

$$\frac{\beta(1-\delta+\alpha k_{1,t+1}^{\alpha-1}(A_{1,t+1}n_{1,t+1})^{1-\alpha})}{\beta(1-\delta+\alpha k_{1,t}^{\alpha-1}(A_{1,t}n_{1,t})^{1-\alpha})}=1$$

Which implies

$$\frac{A_{1,t+1}}{A_{1,t}}=\frac{n_{1,t}}{n_{1,t+1}}\frac{k_{1,t+1}}{k_{1,t}}=(1+g)$$

We also know

$$\begin{aligned}(1+g_{c_1})=\frac{c_{1,t+1}}{c_{1,t}}&=\left(\frac{A_{1,t+1}}{A_{1,t}}\frac{A_{2,t}}{A_2,t+1}\right)^{1-\alpha}\frac{c_{2,t+1}+\bar{c}}{c_{2,t}+\bar{c}}\\&=((1+g)\frac{1}{(1+g)})^{1-\alpha}\frac{c_{2,t+1}+\bar{c}}{c_{2,t}+\bar{c}}\\&=\frac{c_{2,t+1}+\bar{c}}{c_{2,t}+\bar{c}}\end{aligned}$$

Now define g_{c_2} as the growth rate of c_2. By its definition

$$\begin{aligned}1+g_{c_1}&=\frac{c_{2,t}(1+g_{c_2})+\bar{c}}{c_{2,t}+\bar{c}}\\&=\frac{c_{2,t}(1+g_{c_2})+\bar{c}+\bar{c}g_{c_2}-\bar{c}g_{c_2}}{c_{2,t}+\bar{c}}\\&=\frac{(c_{2,t}+\bar{c})(1+g_{c_2})-\bar{c}g_{c_2}}{c_{2,t}+\bar{c}}\\&=1+g_{c_2}-\frac{\bar{c}g_{c_2}}{c_{2,t}+\bar{c}}\end{aligned}$$

Which implies that

$$g_{c_2}=\frac{g_{c_1}}{1-\frac{\bar{c}}{c_{2,t}+\bar{c}}}$$

Since $\bar{c}$ and $c_{2,t}$ are positive this implies that $g_{c_2}>g_{c_1}$. Therefore on any BGP consumption of the second good is increasing relative to the first good.

Output for each good will grow by,

$$\begin{aligned}\frac{y_{1,t+1}}{y_{1,t}}&=\left(\frac{k_{1,t+1}}{k_{1,t}}\right)^{\alpha}\left(\frac{A_{1,t+1}}{A_{1,t}}\right)^{1-\alpha}\left(\frac{n_{1,t+1}}{n_{1,t}}\right)^{1-\alpha}\\&=(1+g)^{\alpha}(1+g)^{1-\alpha}\left(\frac{n_{1,t+1}}{n_{1,t}}\right)\\&=(1+g)\frac{n_{1,t+1}}{n_{1,t}}\end{aligned}$$

Similarly for the second good. Therefore, since labor is fixed over time and workers may only shift between industries, the economy will grow by $(1+g)$.

Part (d)

When $g_1 < g_2$, the sector with higher productivity (i.e., sector 2) produces relatively more output over time, reinforcing the preference-induced structural change from the previous part. We see that

$$\frac{c_{1,t}}{c_{2,t} + \bar{c}} = \frac{a}{1-a}\left(\frac{A_{1,t}}{A_{2,t}}\right)^{1-\alpha}$$

Therefore as $A_{2,t}$ increases relative to $A_{1,t}$ over time, $c_1 1$ decreases relative to c_{2t}. In the limit $c_{1,t} \to 0$.

Part (e)

Going back to part (c) where we assumed that we were on a balanced growth path and $\frac{c_{1,t+1}}{c_{1,t}}$ is constant, suppose it wasn't and we perturbed the system. Then we would have an inequality between productivity growth and capital evolution over time. Capital choice in the next period would immediately begin to adjust adjust sending us back to the steady state. This would occur regardless of the direction of the perturbation and direction of the ineuqality.

Problem 5

See the solutions manual to Ljungqvist and Sargent (it's easy to find).

Problem 6

Part (a)

A competitive equilibrium is a sequence of allocations $\{c_t^W, c_t^C, k_t, x_t\}_{t=0}^{\infty}, \{y_t, k_t, n_t\}_{t=0}$ and prices $\{p_t^c, p_t^x, w_t, r_t\}_{t=0}^{\infty}$ such that:

- Workers solve:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

$$s.t. \quad p_t c_t \leq w_t \times 1 \; \forall t$$

$$c_t \geq 0 \; \forall t$$

$$\forall t$$

- Capitalists maximize their own utility while operating production, selling consumption good and buying labor, and deciding optimal accumulation of capital. Their budget constraint shows that whatever is left from production after personal consumption and investment, must be sufficient to pay for the wages.

$$\max_{\{c_t,k_{t+1},x_t,n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{c_t^{1-\sigma}}{1-\sigma}$$
$$\begin{aligned} s.t. \quad & w_t n_t \leq p_t(z_t k_t^{\alpha} n_t^{\beta} - c_t - x_t) \ \forall t \\ & c_t, x_t \geq 0 \ \forall t \\ & k_{t+1} = k_t(1-\delta) + x_t \ \forall t \\ & k_0 \text{ given} \end{aligned}$$

Or equivalently, getting rid of x_t:

$$\max_{\{c_t,k_{t+1},n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{c_t^{1-\sigma}}{1-\sigma}$$
$$\begin{aligned} s.t. \quad & w_t n_t \leq p_t(z_t k_t^{\alpha} n_t^{\beta} - c_t - k_{t+1} + k_t(1-\delta)) \ \forall t \\ & c_t \geq 0 \ \forall t \\ & k_{t+1} - k_t(1-\delta) \geq 0 \ \forall t \\ & k_0 \text{ given} \end{aligned}$$

You could also assume that the consumption good and the investment good are not the same good and they are produced in separate plants with separate stocks of capital and labor demands, which are both chosen optimally. It will not make a difference *in the long run*, meaning at the steady state or BGP, which is all we are looking at in this problem. So better keep it simple.

- Markets clear $\forall t$:

$$\begin{aligned} c_t^C + c_t^W &= y_t = z_t(k_t)^{\alpha}(n_t)^{\beta} \\ n_t &= 1 \end{aligned}$$

Part (b)

Capitalist's Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{c_t^{1-\sigma}}{1-\sigma}$$
$$+ \sum_{t=0}^{\infty} \lambda_t \left\{ p_t(z_t k_t^\alpha n_t^\beta - c_t - k_{t+1} + k_t(1-\delta)) - w_t n_t \right\} \quad + \sum_{t=0}^{\infty} \mu_t \left\{ k_{t+1} - k_t(1-\delta) \right\} + \sum_{t=0}^{\infty} \kappa_t(c_t)$$

At optimum $\kappa_t = 0$ since consumption is always positive, $\mu_t = 0$ since it's never optimal to invest nothing. Hence the FOC from the capitalist's problem are:

$$FOC(k_{t+1}) \ : \lambda_t p_t = \lambda_{t+1} p_{t+1} (z_{t+1} n_{t+1}^\beta \alpha k_{t+1}^{\alpha-1} + (1-\delta))$$
$$FOC(c_t) \ : \tilde{\beta}^t c_t^{-\sigma} = \lambda_t p_t$$
$$FOC(n_t) \ : p_t z_t k_t^\alpha \beta n_t^{\beta-1} = w_t$$

Rearranging the latter we get the labor demand:

$$n_t = [w_t / (p_t z_t k_t^\alpha \beta)]^{\frac{1}{\beta-1}}$$

Imposing market clearing wages are determined:

$$n_t^* = 1 = [w_t^* / (p_t z_t k_t^\alpha \beta)]^{\frac{1}{\beta-1}}$$
$$w_t^* = p_t z_t k_t^\alpha \beta$$

The resulting Lagrangian for the capitalists is (keeping in mind that they are still assumed to be price takers):

$$\mathcal{L} = \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left\{ p_t(z_t k_t^\alpha \times 1 - c_t - k_{t+1} + k_t(1-\delta)) - p_t z_t k_t^\alpha \beta \times 1 \right\}$$
$$= \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \left\{ p_t(z_t k_t^\alpha (1-\beta) - c_t - k_{t+1} + k_t(1-\delta)) \right\}$$

Part (c)

The recursive form of this problem is as follows (noticing that we dropped prices in the constraint):

$$V(k_t) = \max_{k_{t+1}} \frac{c_t^{1-\sigma}}{1-\sigma} + \tilde{\beta} V(k_{t+1})$$
$$s.t. \quad c_t = z_t k_t^\alpha (1-\beta) - k_{t+1} + k_t(1-\delta))$$
$$k_{t+1} - k_t(1-\delta) \geq 0 \ \forall t$$
$$c_t \geq 0$$

The fixed point of this functional equation is:

- Unique: SL Theorem 4.6.
  - $k_{t+1} \in \mathbb{R}$, a convex set
  - The constraints on k_{t+1} define a nonempty, compact-valued, and continuous correspondence from k_t
  - Returns are bounded due to time discounting and continuous production function on compact set
  - Hence the RHS on the functional equation is a contraction and has a unique fixed point
- Increasing: SL Theorem 4.7
  - The flow payoff is increasing in the state variable:
    You can see directly that $\frac{\partial c_t}{\partial k_t > 0}$ and the utility is increasing in c_t too.
  - The constraints can be rewritten as:
    $$z_t k_t^\alpha (1-\beta) + k_t(1-\delta)) \geq k_{t+1} \geq k_t(1-\delta))$$
    The theorem requires that this correspondence $\Gamma(k_t)$ be monotone in the sense $k \leq k' \implies \Gamma(k) \subseteq \Gamma(k')$. This is not the case here (since we rule out divestment) but by strict increasingness of the flow payoff and assuming that V is at least nondecreasing, it is sufficient to show that $\max\{k_{t+1} \mid z_t k_t^\alpha(1-\beta) + k_t(1-\delta)) \geq k_{t+1} \geq k_t(1-\delta))\} > \max\{k_{t+1} \mid z_t k_t'^\alpha(1-\beta) + k_t'(1-\delta)) \geq k_{t+1} \geq k_t'(1-\delta))\}$ for $k_t' > k_t$, which is indeed the case because the LHS of the first inequality is strictly increasing in k. By strict increasingness of the flow payoff (call it F) and weak increasingness of the solution V, we infer the *strict* increasingness of V.
- Concave: SL Theorem 4.8:
  - The flow payoff is strictly concave since the utility function is strictly concave for $\sigma > 1$ and c is concave both in k_t and k_{t+1} since it's a sum of concave functions in these arguments.
  - The choice set defined by the constraints is a convex correspondence:
    $$z_t \tilde{k}_t^\alpha (1-\beta) + \tilde{k}_t(1-\delta)) \geq \tilde{k}_{t+1} \geq \tilde{k}_t(1-\delta))$$
    $$z_t \bar{k}_t^\alpha (1-\beta) + \bar{k}_t(1-\delta)) \geq \bar{k}_{t+1} \geq \bar{k}_t(1-\delta))$$

taking a convex combination:

$$z_t(\theta\tilde{k}_t^\alpha + (1-\theta)\bar{k}_t^\alpha)(1-\beta) + (\theta\tilde{k}_t + (1-\theta)\bar{k}_t)(1-\delta))$$
$$\geq (\theta\tilde{k}_{t+1} + (1-\theta)\bar{k}_{t+1})(1-\delta)) \geq (\theta\tilde{k}_t + (1-\theta)\bar{k}_t)(1-\delta))$$

but by concavity of the production function we have

$$z_t(\theta\tilde{k}_t + (1-\theta)\bar{k}_t)^\alpha(1-\beta) + (\theta\tilde{k}_t + (1-\theta)\bar{k}_t)(1-\delta))$$
$$\geq z_t(\theta\tilde{k}_t^\alpha + (1-\theta)\bar{k}_t^\alpha)(1-\beta) + (\theta\tilde{k}_t + (1-\theta)\bar{k}_t)(1-\delta))$$
$$\geq (\theta\tilde{k}_{t+1} + (1-\theta)\bar{k}_{t+1})(1-\delta)) \geq (\theta\tilde{k}_t + (1-\theta)\bar{k}_t)(1-\delta))$$

- Then the solution $V(k_t)$ is strictly concave

- Differentiable: SL Theorem 4.11
  - The objective function is indeed continuosly differentiable for all values in the interior of the constraints. Hence our solution $V(k_t)$ is continuously differentiable.

Part (d)

Redefining all variables by taking the ratio with the TFP z_t we can find the steady state of the new system, which corresponds to the BGP in the original system. The constraints and the objective function will need to be adjusted accordingly. Mind that the steady state we are going to find is bound to be somewhat "partial" because labor supply does not grow at all in levels. Hence it's consumption, capital, and output that will be constant in the new system.

1. Start by reformulating the constraints. This will make immediately clear what you should divide all your variables for. Ultimately, you want to get rid of those terms that change in time—z_t in our case. The trick is to redefine our TFP to be labor-augmenting by defining $\tilde{z}_t^\beta = z_t$ and then divide our constraint by $\tilde{z}_t$. Notice that now $\tilde{z}_{t+1} = g^{1/\beta}\tilde{z}_t = \tilde{g}\tilde{z}_t$ and keep in mind that *we need to assume* $\alpha + \beta = 1$ for this to work: we need HD1.

$$c_t/\tilde{z}_t = \frac{\tilde{z}_t^\beta}{\tilde{z}_t}k_t^\alpha(1-\beta) - \frac{k_{t+1}}{\tilde{z}_{t+1}}\tilde{g} + k_t/\tilde{z}_t(1-\delta))$$
$$\frac{k_{t+1}}{\tilde{z}_{t+1}}\tilde{g} - k_t/\tilde{z}_t(1-\delta) \geq 0 \;\forall t$$
$$c_t/\tilde{z}_t \geq 0$$

$$\tilde{c}_t = \tilde{k}_t^{\alpha}(1-\beta) - \tilde{g}\tilde{k}_{t+1} + \tilde{k}_t(1-\delta))$$
$$\tilde{g}\tilde{k}_{t+1} - \tilde{k}_t(1-\delta) \geq 0 \ \forall t$$
$$\tilde{c}_t \geq 0$$

2. Reformulate the utility: you need to multiply and divide the flow payoff by $\tilde{z}_t^{1-\sigma}$ to express utlity in the new variable $\tilde{c}_t = \frac{c_t}{\tilde{z}_t}$. Then remember that $\tilde{z}_t = \tilde{z}_0\tilde{g}^t$. Finally, dropping $\tilde{z}_0$ will not change the maximization problem.

Hence the reformulated capitalist's problem:

$$V(\tilde{k}_t) = \max_{\tilde{k}_{t+1}} \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma} + \tilde{\beta}\tilde{g}^{1-\sigma}V(\tilde{k}_{t+1})$$
$$s.t. \quad \tilde{c}_t = \alpha\tilde{k}_t^{\alpha} - \tilde{g}\tilde{k}_{t+1} + \tilde{k}_t(1-\delta)$$
$$\tilde{g}\tilde{k}_{t+1} - \tilde{k}_t(1-\delta) \geq 0 \ \forall t$$
$$\tilde{c}_t \geq 0$$

$$\text{Take FOC w.r.t. } \tilde{k}_{t+1}: \quad \tilde{c}_t^{-\sigma}(-\tilde{g}) + \tilde{\beta}\tilde{g}^{1-\sigma}V'(\tilde{k}_{t+1}) = 0$$
$$\text{Use Envelope Theorem to get } V'(\tilde{k}_{t+1}): \quad V'(\tilde{k}_{t+1}) = \tilde{c}_{t+1}^{-\sigma}(\alpha^2\tilde{k}^{\alpha-1} + 1 - \delta)$$
$$\text{Plug in}: \quad \tilde{c}_t^{-\sigma}\tilde{g} = \tilde{\beta}\tilde{g}^{1-\sigma}\tilde{c}_{t+1}^{-\sigma}(\alpha^2\tilde{k}^{\alpha-1} + 1 - \delta)$$

The Euler equation is:

$$\left(\frac{\tilde{c}_{t+1}}{\tilde{c}_t}\right)^{\sigma} = \tilde{\beta}\tilde{g}^{-\sigma}[\alpha^2\tilde{k}_t^{\alpha-1} + 1 - \delta]$$

Impose steady state:

$$\left(\frac{\tilde{c}}{\tilde{c}}\right)^{\sigma} = 1 = \tilde{\beta}\tilde{g}^{-\sigma}[\alpha^2\tilde{k}^{\alpha-1} + 1 - \delta]$$
$$\tilde{k} = \left[\frac{\tilde{g}^{\sigma} - \tilde{\beta}(1-\delta)}{\tilde{\beta}\alpha^2}\right]^{\frac{1}{1-\alpha}}$$
$$\implies k_t = \left[\frac{\tilde{g}^{\sigma} - \tilde{\beta}(1-\delta)}{\tilde{\beta}\alpha^2}\right]^{\frac{1}{1-\alpha}} \times \tilde{z}_0\tilde{g}^t$$
$$\implies \frac{k_{t+1}}{k_t} = \tilde{g} = g^{\frac{1}{1-\alpha}}$$

The capital-output ratio:

$$\frac{k_t}{y_t} = \frac{k_t}{z_t k_t^\alpha n_t^{1-\alpha}} = \frac{k_t}{z_t k_t^\alpha \times 1}$$

Now transform the denominator as before

$$= \frac{k_t/\tilde{z}_t}{\tilde{z}_t^{1-\alpha}/\tilde{z}_t k_t^\alpha \times 1} = \frac{\tilde{k}_t}{\tilde{k}_t^\alpha}$$

Capital- output ratio remains constant.

Real wages.

$$\frac{w_t^\star}{p_t} = z_t k_t^\alpha (1-\alpha) = \tilde{z}_t^{1-\alpha} k_t^\alpha (1-\alpha) = \tilde{z}_t \tilde{k}_t^\alpha (1-\alpha)$$

And their growth rates on the BGP path (use $\tilde{k}_{t+1} = \tilde{k}_t$):

$$\frac{w_t^\star}{p_t} / \frac{w_{t+1}^\star}{p_{t+1}} = \frac{\tilde{z}_{t+1}\tilde{k}_{t+1}(1-\alpha)}{\tilde{z}_t \tilde{k}_t (1-\alpha)} = \tilde{g} = g^{\frac{1}{1-\alpha}}$$

Part (e)

The return on capital is usually the RHS of the Euler equation:

$$\left(\frac{\tilde{c}_{t+1}}{\tilde{c}_t}\right)^\sigma = \tilde{\beta}\tilde{g}^{-\sigma}[\alpha^2 \tilde{k}_t^{\alpha-1} + 1 - \delta]$$

With a caveat though! In this case, this is not quite the Euler equation we need since is expressed in terms of $\tilde{c}_t$, while agents consume c_t. Convert it back:

$$\left(\frac{\tilde{c}_{t+1}}{\tilde{c}_t}\right)^\sigma = \left(\frac{c_{t+1}/(\tilde{z}_t\tilde{g})}{c_t \tilde{z}_t}\right)^\sigma = \left(\frac{c_{t+1}}{c_t}\right)^\sigma \tilde{g}^{-\sigma} = \tilde{\beta}\tilde{g}^{-\sigma}[\alpha^2 \tilde{k}_t^{\alpha-1} + 1 - \delta]$$

$$\implies \left(\frac{c_{t+1}}{c_t}\right)^\sigma = \tilde{\beta}[\alpha^2 \tilde{k}_t^{\alpha-1} + 1 - \delta]$$

And since $\tilde{k}_t$ is constant on the BGP, we have that the return on capital is constant on the BGP. And since:

$$\tilde{k} = \left[\frac{\tilde{g}^\sigma - \tilde{\beta}(1-\delta)}{\tilde{\beta}\alpha^2}\right]^{\frac{1}{1-\alpha}} = \left[\frac{g^{\frac{\sigma}{1-\alpha}} - \tilde{\beta}(1-\delta)}{\tilde{\beta}\alpha^2}\right]^{\frac{1}{1-\alpha}}$$

We can see that the return on capital is a non-linear, increasing transformation of the BGP growth rate $\tilde{g}$

Part (f)

As mentioned in the previous point, the decline of g decreases the BGP return on capital.

From the capitalist constraint and market clearing we know that the workers' consumption is such that:

$$\begin{aligned}
\tilde{c}^W + \tilde{c}^C &= \tilde{k}^\alpha \\
(1+\xi)\tilde{c}^C &= \tilde{k}^\alpha \\
(1+\xi)[\alpha\tilde{k}^\alpha - \tilde{g}\tilde{k} + \tilde{k}(1-\delta)] &= \tilde{k}^\alpha \\
\xi &= \tilde{k}^\alpha[\alpha\tilde{k}^\alpha - \tilde{g}\tilde{k} + \tilde{k}(1-\delta)]^{-1} - 1 \\
&= [\alpha + \tilde{k}^{1-\alpha}(1-\delta-\tilde{g})]^{-1} - 1
\end{aligned}$$

and since $\partial\tilde{g}/\partial g > 0$ and

$$\begin{aligned}
\frac{\partial}{\partial\tilde{g}}\tilde{k}^{1-\alpha}(1-\delta-\tilde{g}) &= \frac{\partial\tilde{k}^{1-\alpha}}{\partial\tilde{g}}(1-\delta-\tilde{g}) - \tilde{k}^{1-\alpha} \\
&= \frac{1}{1-\alpha}\left[\frac{\tilde{g}-\beta(1-\delta)}{\beta\alpha^2}\right]^{\frac{-\alpha}{1-\alpha}} \frac{-\sigma}{\beta\alpha^2}\tilde{g}^{-\sigma-1} - \tilde{k}^{1-\alpha} < 0
\end{aligned}$$

then $\partial\xi/\partial g > 0$ and an increase in productivity improves the consumption of workers with respect to the capitalists'.

Problem 7

Define $g_f = \frac{\dot{f}}{f} = \frac{\partial}{\partial t}f$.

We have to prove that there exists a BGP if and only if the production function can be written in a form such that prouctivity growth (or some function of it) is labor-augmenting.

$\boxed{\Longleftarrow}$: If there exists an equivalent production function in which productivity growth is labor-augmenting then there exists a BGP.

Refer to the notes to see that reformulating the system taking the ratio of all variables with population and productivity (those variables that are growing exogenously) yields a system that has a steady state, and hence a BGP for the original system.

$\boxed{\Longrightarrow}$: If we have a BGP then there exists a production function with labor-augmenting productivity growth.

Lemma

For any constant α, $g_{X^\alpha Y} = \alpha g_x + g_Y$.

Proof: For any $Z(t)$:

$$
\begin{aligned}
g_Z &= \frac{\dot{Z}}{Z} = \frac{\partial \ln Z(t)}{\partial t} \\
\implies g_{X^\alpha Y} &= \frac{\partial}{\partial t} \ln(X(t)^\alpha Y(t)) \\
&= \alpha \frac{\partial \ln(X(t))}{\partial t} + \frac{\partial \ln(Y(t))}{\partial t} \\
&= \alpha g_X + g_Y
\end{aligned}
$$

The proof is in two parts:

1) Show that on the BGP, output, consumption, and capital all grow at the same rate. We need to assume continuous time and that at some time T the economy reaches the BGP and it stays on that path $\forall t \geq T$

2) Show that homogeneity of degree 1 in the production function with respect to K_t and N_t imply the existence of an equivalent function in which productivity growth is labor-augmenting.

Part(1)

Law of motion of capital:

$$
\begin{aligned}
\dot{K}(t) &= Y(t) - C(t) - \delta K(t) \\
&= I(t) - \delta K(t) \\
g_K K(t) &= I(t) - \delta K(t) \\
K(t)(g_K + \delta) &= I(t) \\
g_k + \delta &= \frac{I(t)}{K(t)}
\end{aligned}
$$

Since the LHS is a constant $\forall t \geq T$, then the RHS is a constant as well, which means that $g_I = g_K$. Specifically, the growth rate of $\frac{I(t)}{K(t)}$ is 0 and by the lemma with $\alpha = -1$:

$$-g_K + g_I = 0 \implies g_I = g_K$$

From feasibility:

$$\begin{aligned} Y(t) &= C(t) + I(t) \\ \dot{Y}(t) &= \dot{C}(t) + \dot{I}(t) \\ \frac{\dot{Y}(t)}{Y(t)} &= \frac{\dot{C}(t)}{Y(t)} + \frac{\dot{I}(t)}{Y(t)} \\ &= \frac{\dot{C}(t)}{C(t)}\frac{C(t)}{Y(t)} + \frac{\dot{I}(t)}{I(t)}\frac{I(t)}{Y(t)} \\ &= g_C\frac{C(t)}{Y(t)} + g_I\frac{I(t)}{Y(t)} \\ &= g_C\frac{C(t)}{Y(t)} + g_I\left(1 - \frac{C(t)}{Y(t)}\right) \\ g_Y &= (g_C - g_I)\frac{C(t)}{Y(t)} + g_I \end{aligned}$$

since on the BGP g_Y, g_K, g_I are constants, then $C(t)/Y(t)$ is a constant too and like before it implies $g_C = g_Y$

Using feasibility again but substituting $\frac{C(t)}{Y(t)} = 1 - \frac{I(t)}{Y(t)}$ gives $g_I = g_K$. Hence

$$g_Y = g_C = g_k$$

Part (2)

We are given the production function $F(K(t), N(t), z(t)) = Y(t)$. By homogeneity of degree 1 in $K(t)$ and $N(t)$, we have:

$$\begin{aligned} Y(T) &= F(K(T), N(T), z(T)) \\ Y(T)\frac{Y(t)}{Y(T)} &= \frac{Y(t)}{Y(T)}F(K(T), N(T), z(T)) \\ Y(T)\frac{Y(t)}{Y(T)} &= F(K(T)\frac{Y(t)}{Y(T)}, N(T)\frac{Y(t)}{Y(T)}, z(T)) \end{aligned}$$

But since $\forall t \geq T$ we have $g_Y = g_K$ then $\frac{Y(t)}{Y(T)} = \frac{K(t)}{K(T)}$. Hence:

$$\begin{aligned} Y(T)\frac{Y(t)}{Y(T)} &= F(K(T)\frac{K(t)}{K(T)}, N(T)\frac{Y(t)}{Y(T)}, z(T)) \\ Y(t) &= F(K(t), N(T)\frac{Y(t)}{Y(T)}, z(T)) \end{aligned}$$

Now define $\forall t \geq T$ $\tilde{z}(t) = \frac{Y(t)}{N(t)}\frac{N(T)}{Y(T)}$ and define $\tilde{F}(K(t), \tilde{z}(t)N(t)) = F\left(K(t), N(t)\tilde{z}(t), z(\underline{T})\right)$, which is the same function having fixed the third argument to the value $z(T)$. Now to prove that the newly defined function yields the desired production level $Y(t)$ given the inputs $K(t)$ and $N(t)$, just expand $\tilde{z}(t)$ on the RHS:

$$\begin{aligned}
\tilde{F}(K(t), \tilde{z}(t)N(t)) &= F\left(K(t), N(t)\tilde{z}(t), z(\underline{T})\right) \\
&= F\left(K(t), N(t)\frac{Y(t)}{N(t)}\frac{N(T)}{Y(T)}, z(\underline{T})\right) \\
&= F\left(K(t), N(T)\frac{Y(t)}{Y(T)}, z(T)\right) \\
&= Y(t)
\end{aligned}$$

An alternative proof can be found in Chapter 2 of Acemoglu's book.

Problem 8

Part (a)

A competitive equilibrium is defined as a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, $\{c_t^{\tau i}, x_t^{\tau i}, k_t^{\tau i}, n_t^{\tau i}\}_{t=0}^{\infty}$, $\{y_t, n_t, k_t\}_{t=0}^{\infty}$ (where t indexes time, τ indexes date of birth, and i indexes individual) such that Households solve:

$$\begin{aligned}
\max_{\{c_t^{\tau i}, x_t^{\tau i}, k_t^{\tau i}, n_t^{\tau i}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u_i(c_t^{\tau i}) \\
s.t. \quad & \sum_{t=0}^{\infty} p_t(c_t^{\tau i} + x_t^{\tau i}) \leq \sum_{t=0}^{\infty} (r_t k_t^{\tau i} + w_t n_t^{\tau i}) \\
& k_{t+1}^{\tau i} = (1-\delta)k_t^{\tau i} + x_t^{\tau i} \\
& k_\tau^{\tau i} \text{ given}
\end{aligned}$$

Firms solve:

$$\max_{\{y_t\}_{t=0}^{\infty}} p_t F(K_t, N_t) - r_t K_t - w_t N_t$$

Markets clear:

$$\begin{aligned}
&\sum_{\tau=0}^{t} \int_{N_{\tau-1}}^{N_\tau} k_t^{\tau i} di = K_t \quad \text{assuming } N_{-1} = 0 \\
&\sum_{\tau=0}^{t} \int_{N_{\tau-1}}^{N_\tau} c_t^{\tau i} di + \sum_{\tau=0}^{t} \int_{N_{\tau-1}}^{N_\tau} x_t^{\tau i} di = C_t + X_t = F(K_t, N_t)
\end{aligned}$$

Part (b)

The easiest way to have the same consumption for each household alive at time t, independently from their birth date, is to have homogeneous agents within each period t. That is, they have identical preferences and budget constraints from t onwards[1]. This includes time discounting, which will have to be the same despite different agents being born at different times:

Objective function for household born at t: $\max_{\{c_t^{\tau i}, x_t^{\tau i}, k_t^{\tau i}, n_t^{\tau i}\}_{t=0}^{\infty}} \sum_{s=0}^{\infty} 1\{s \geq t\} \beta^s u(c_s^i) = \max_{\{c_s^i, k_{s+1}^i\}} \sum_{s=t}^{\infty} \beta^s u(c_s^i)$

The newborn generation at every t, who could not accumulate capital in previous periods will also have the same budget constraint as everybody else. This means that their endowments of capital are equal to the stock of capital of any other agent alive at t. On the BGP, this corresponds to endowing agents with the BGP per-capita capital itself: some level $\bar{k}$.

The idea is that homogenous agents at time t solve the same maximization problem and hence choose the same consumption.

Part (c)

Under the assumptions of Part (b), the CE is such that consumption is the same for all agents living at any time t. To find what welfare weights support the same allocation as a result of the planner's problem, we refer to the notes and make the due changes. Notice that the difference with respect to the case presented in the notes is the welfare of each agent. Instead of no discounting at the moment of birth t, in this case agents discount by a factor of β^t.

The planner's objective function:

$$\sum_{t=0}^{\infty} \int_{N_{t-1}}^{N_t} g_t(i) \sum_{s=t}^{\infty} \beta^s u(c_t^i)$$
$$\sum_{t=0}^{\infty} \sum_{s=t}^{\infty} \beta^s u(c_t^i) \int_{N_{t-1}}^{N_t} g_t(i) = \sum_{t=0}^{\infty} \sum_{s=t}^{\infty} \beta^s u(c_t^i) \hat{g}_t$$

[1] Although I can't exclude that under certain conditions it is possible to trade off heterogenous preferences and budget constraints so that agents happen to choose exactly the same consumption levels.

...since individuals within a generation are given the same consumption at each period.

$$\sum_{t=0}^{\infty}\sum_{s=t}^{\infty}\beta^s u(c_t^i)\hat{g}_t = u(c_0^0)\hat{g}_0 + \beta[u(c_1^0)\hat{g}_0 + u(c_1^1)\hat{g}_1] + \beta^2[u(c_2^0)\hat{g}_0 + u(c_2^1)\hat{g}_1 + u(c_2^2)\hat{g}_2] + ...$$

(where c_0^1 means consumption at time 0 for any consumer born at time 1)

$$= u(c_0)\hat{g}_0 + \beta[u(c_1)\hat{g}_0 + u(c_1)\hat{g}_1] + \beta^2[u(c_2)\hat{g}_0 + u(c_2)\hat{g}_1 + u(c_2)\hat{g}_2] + ...$$

(since all individuals alive at time t get the same consumption)

$$= \sum_{t=0}^{\infty}\beta^t u(c_t)\sum_{s=0}^{t}\hat{g}_t$$

Now, if $g_t(i) = 1\ \forall i, t$ then $\hat{g}_t = N_t - N_{t-1}$ and $beta^t \sum_{s=0}^{t}\hat{g}_t = \beta^t N_t$, resulting in the social welfare function:

$$\sum_{t=0}^{\infty}\beta^t N_t u(c_t)$$