Introduction to Macro

- Perhaps the most important question that we care about in macro: Distribution of income: across geographic areas, individuals, how it changes over time, income shares, etc.

- Striking observation: a lot of regularities in the past one or two centuries - over long horizons:

  – Kaldor Facts:

    * Labor productivity has grown at a sustained rate in developed economies
    * Capital per worker has also grown
    * Real interest rate has been stable
    * Capital-output ratio has been stable
    * Factor shares have been stable
    * growth rates of labor productivity highly variable across countries

  – Recent arguments and observations by Piketty (2014), or Karabarbounis and Neiman (2014)

  – Convergence and cross-country observations

- Certain regularities over short periods of time:

  – Burns and Mitchell (1946) "facts":

    * $\sigma_{\text{Consumption-nondurable}} \approx \sigma_{\text{Output}}$
    * $\sigma_{\text{Consumption-durable}} > \sigma_{\text{Output}}$
    * $\sigma_{\text{investment}} \approx 3\sigma_{\text{Output}}$
    * $\sigma_{\text{Government purchases}} < \sigma_{\text{output}}$
    * $\sigma_{\text{Total hours workers}} \approx \sigma_{\text{output}}$; $\sigma_{\text{hours worked per worker}} \ll \sigma_{\text{output}}$; $\sigma_{\text{Employment}} \approx \sigma_{\text{output}}$
    * $\sigma_{\text{capital}} \ll \sigma_{\text{output}}$

- How can we even think about these issues? Need a theoretical framework.

- Old Macro: specify equations - Lucas Critique

- Modern macro: build from bottom up: specify agents with preferences, technology of production, prices and general equilibrium

- Standard Macro Model:

  – Time: $t = 0, 1, \cdots$

  – Commodities: capital, output (or consumption good), leisure/labor and investment good in each period -infinitely many commodities

  – Firms:

    * producers of consumption goods: $j = 1, \cdots, J_c$, production is given by $y^j_{t,c} = F^c_{j_c,t}\left(k^j_c, n^j_c\right)$

    - producers of investment goods: $j = 1, \cdots, J_k$, production is given by $y_{t,k}^j = F_{j_k,t}^k \left(k_k^j, n_k^j\right)$
- Households: $j \in \{0, \cdots, J\}$ preference over consumption, leisure, investment $U^i \left(\mathbf{z}^i\right)$ with $\mathbf{z}^i = \left(\{c_t^i, \ell_t^i, x_t^i, k_t^i\}_{t=0}^{\infty}\right)$, own capital and labor. More formally, U^j is a function from ℓ_4^p (space of infinite sequences on $\mathbb{R}$ equipped with an appropriate norm say $d(x, y) = \sum_{n=0}^{\infty} |x_n - y_n|^p$) into $\mathbb{R}$ and is continuous with respect to these sequenes
- Endowments:
    - endowment of leisure: e_t^i
    - time-0 capital: k_{-1}^i
    - shares in firms: $\theta_{i,c}^j, \theta_{i,k}^j$ with

$$\sum_{i=1}^{I} \theta_{i,c}^j = 1, \sum_{i=1}^{I} \theta_{i,c}^j = 1$$

- prices and markets:
    - time-0 trading: trading is done before any consumption or investment takes place; this is sort of weird but you will show in your homework that it will be equivalent to more reasonable market structure
    - market for consumption good at t: price is given by $p_{c,t}$
    - market for capital goods at t: price is given by $p_{x,t}$
    - market for renting out labor at t: price is given by w_t
    - market for renting of capital at t: price is given by r_t
- law of motion for capital:

$$k_{t+1}^i = k_t^i (1 - \delta) + x_t^i$$

  where x_t^i is the amount of investment goods purchased by household i
- Markets have to clear or feasibility constraints have to hold

$$\sum_{i=1}^{I} c_t^i = \sum_{j=1}^{J_c} y_{t,c}^j$$

$$\sum_{i=1}^{I} x_t^i = \sum_{j=1}^{J_k} y_{t,k}^j$$

$$\sum_{i=1}^{I} k_t^i = \sum_{j=1}^{J_c} k_{t,c}^j + \sum_{j=1}^{J_k} k_{t,k}^j$$

$$\sum_{i=1}^{I} e_t^i - \ell_t^i = \sum_{j=1}^{J} n_{t,c}^j + \sum_{j=1}^{J_k} n_{t,k}^j$$

- Definition of competitive equilibrium: sequence of allocations $\hat{\mathbf{z}}^i = \left\{\hat{c}_t^i, \hat{\ell}_t^i, \hat{k}_t^i, \hat{x}_t^j\right\}_{t=0,\cdots,\infty, j=1,\cdots,J}$, $\left\{\hat{y}_{t,c}^j, \hat{k}_{t,c}^j, \hat{n}_{t,c}^j\right\}_{j=1,\cdots,J_c, t=0,\cdots,\infty}$, $\left\{\hat{y}_{t,k}^j, \hat{k}_{t,k}^j, \hat{n}_{t,k}^j\right\}_{j=1,\cdots,J_k, t=0,\cdots,\infty}$ and prices $\{\hat{p}_{c,t}, \hat{p}_{x,t}, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$ constitutes a competitive equilibrium such that:

* Households maximize taking prices and firms' profits as given. That is they solve the following optimization

$$\hat{\mathbf{z}}^i \in \arg\max_{z^i} U^i\left(\mathbf{z}^i\right)$$

subject to

$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}c_t^i + \hat{p}_{x,t}x_t^i\right] \leq \sum_{t=0}^{\infty}\left[\hat{w}_t\left(e_t^i - \ell_t^i\right) + \hat{r}_t k_t^i\right] + \Pi^i$$
$$k_{t+1}^i = k_t^i\left(1-\delta\right) + x_t^i$$
$$c_t^i, x_t^i, \ell_t^i, k_t^i \geq 0$$
$$k_0^i \text{ :given}$$

where Π^i is total profits from firms.

* Firms maximize taking prices as given. They solve the following optimization problems
    · Consumption good producers

$$\left\{\hat{k}_t^j, \hat{n}_t^j\right\} \in \arg\max_{k,n} \hat{p}_{c,t}F_{t,c}^j\left(k,n\right) - \hat{w}_t n - \hat{r}_t k$$

    · Investment good producers

$$\left\{\hat{k}_t^j, \hat{n}_t^j\right\} \in \arg\max_{k,n} \hat{p}_{x,t}F_{t,k}^j\left(k,n\right) - \hat{w}_t n - \hat{r}_t k$$

* Allocations are feasible
* Profits are consistent with firm behavior

$$\Pi^j = \sum_{t=0}^{\infty}\sum_{j=1}^{J_c}\theta_{i,c}^j\left[\hat{p}_{c,t}\hat{y}_{t,c}^j - \hat{w}_t\hat{n}_{t,c}^j - \hat{r}_t\hat{k}_{t,c}^j\right]$$
$$+\sum_{t=0}^{\infty}\sum_{i_k=1}^{J_k}\theta_{i,k}^j\left[\hat{p}_{x,t}\hat{y}_t^j - \hat{w}_t\hat{n}_{t,k}^j - \hat{r}_t\hat{k}_{t,k}^j\right]$$

– It is worth noting that arrangement of ownership and production in this economy is not exactly the same thing that we see in the real world. In reality households own shares in firms and then firms decide about purchasing structures and equipment and investment goods. Here, capital is directly owned by households and is rented out to firms to use for production. It turns out, if we switch to that alternative view of the world, nothing changes. The equilibrium definition is a bit more involved but allocations do not change. This is because of the so-called complete market assumption. That is, the market structure that we have imposed is quite general even though it is weird. Any allocation that is achievable with some other trading mechanism can be also achieved in our time-0 trading mechanism.

- Notation: for simplicity we refer to an allocation with $\mathbf{z}$ which includes consumption, leisure, investment, capital of all households and firm's allocation. We will refer to the set of feasible allocations as $\mathcal{Z}$.
- What is not in this?
  - Finance: no one is borrowing or lending (see homework for this); no one is trading shares of firms
  - Wiggles: will introduce later
  - Government: will introduce later
  - Externalities: will introduce later
  - Rest of the world: probably wont have time to talk about - maybe in the homeworks!
- First Welfare theorem
  - Computing CE is hard; have to solve for infinite sequences of prices and allocations
  - We can use the first welfare theorem to make it easier to solve

Theorem 1. *Suppose that a sequence of allocations* $\left\{\hat{c}_t^j, \hat{\ell}_t^j, \hat{k}_t^j, \hat{x}_t^j\right\}_{t=0,\cdots,\infty, j=1,\cdots,J}$, $\left\{\hat{y}_t^{i_c}, \hat{k}_t^{i_c}, \hat{n}_t^{i_c}\right\}_{i_c=1,\cdots,I_c, t=0,\cdots,\infty}$, $\left\{\hat{y}_t^{i_k}, \hat{k}_t^{i_k}, \hat{n}_t^{i_k}\right\}_{i_k=1,\cdots,I_k, t=0,\cdots,\infty}$ *and prices* $\{\hat{p}_{c,t}, \hat{p}_{x,t}, \hat{r}_t, \hat{w}_t\}_{t=0}^{\infty}$ *constitutes a CE. Suppose further that* $U^j(\cdot)$ *is locally non-satiated*[1] *and that*

$$\sum_{j=1}^{J}\sum_{t=0}^{\infty} \hat{p}_{c,t}\hat{c}_t^j + \hat{p}_{x,t}\hat{x}_t^j < \infty, \forall j = 1, \cdots, J.$$

Then the allocation, $\left\{\hat{c}_t^j, \hat{\ell}_t^j, \hat{k}_t^j, \hat{x}_t^j\right\}_{t=0,\cdots,\infty, j=1,\cdots,J}$, $\left\{\hat{y}_t^{i_c}, \hat{k}_t^{i_c}, \hat{n}_t^{i_c}\right\}_{i_c=1,\cdots,I_c, t=0,\cdots,\infty}$, $\left\{\hat{y}_t^{i_k}, \hat{k}_t^{i_k}, \hat{n}_t^{i_k}\right\}_{i_k=1,\cdots,I_k, t=0,\cdots,\infty}$ *is Pareto optimal.*

Proof. Suppose not. That is, suppose there exists an alternative allocation $\left\{\tilde{c}_t^j, \tilde{\ell}_t^j, \tilde{k}_t^j, \tilde{x}_t^j\right\}_{t=0,\cdots,\infty, j=1,\cdots,J}$, $\left\{\tilde{y}_t^{i_c}, \tilde{k}_t^{i_c}, \tilde{n}_t^{i_c}\right\}_{i_c=1,\cdots,I_c, t=0,\cdots,\infty}$, $\left\{\tilde{y}_t^{i_k}, \tilde{k}_t^{i_k}, \tilde{n}_t^{i_k}\right\}_{i_k=1,\cdots,I_k, t=0,\cdots,\infty}$ which is feasible and satisfies

$$U^j\left(\left\{\tilde{c}_t^j, \tilde{\ell}_t^j\right\}\right) \geq U^j\left(\left\{\hat{c}_t^j, \hat{\ell}_t^j\right\}\right), \forall j = 1, \cdots, J$$

with at least one inequality being strict. Without loss of generality assume that this is household number $j = 1$. Then by definition of competitive equilibrium it must be that

$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\tilde{c}_t^j + \hat{p}_{x,t}\tilde{x}_t^j + \hat{w}\tilde{\ell}_t^j - \hat{r}_t\tilde{k}_t^j\right] \geq \sum_{t=0}^{\infty}\hat{w}_t e_t^j + \hat{\Pi}^j$$
$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\tilde{c}_t^1 + \hat{p}_{x,t}\tilde{x}_t^1 + \hat{w}\tilde{\ell}_t^1 - \hat{r}_t\tilde{k}_t^1\right] > \sum_{t=0}^{\infty}\hat{w}_t e_t^1 + \hat{\Pi}^1$$

[1]Formally a function $f(x)$ that maps any space X (take it to be a Banach space) into $\mathbb{R}$ is locally non-satiated if for any $x \in X$ and $\varepsilon > 0$, there is $x' \in X$ such that $d(x, x') < \varepsilon$ and $f(x') > f(x)$. Informally, it means that the individual does not get satiated at some point.

Question: Why?

Note that also that by optimality of firms' decisions it must be that

$$\hat{\Pi}^j \geq \tilde{\Pi}^j$$

where $\tilde{\Pi}^j$ is the total profits accrued to household j using prices in the competitive equilibrium (Why?). We therefore have

$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\tilde{c}_t^j + \hat{p}_{x,t}\tilde{x}_t^j + \hat{w}\tilde{\ell}_t^j - \hat{r}_t\tilde{k}_t^j\right] \geq \sum_{t=0}^{\infty}\hat{w}_t e_t^j + \tilde{\Pi}^j$$
$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\tilde{c}_t^1 + \hat{p}_{x,t}\tilde{x}_t^1 + \hat{w}\tilde{\ell}_t^1 - \hat{r}_t\tilde{k}_t^1\right] > \sum_{t=0}^{\infty}\hat{w}_t e_t^1 + \tilde{\Pi}^1$$

If we sum over the above, we get

$$\sum_{j=1}^{J}\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\tilde{c}_t^j + \hat{p}_{x,t}\tilde{x}_t^j + \hat{w}\tilde{\ell}_t^j - \hat{r}_t\tilde{k}_t^j\right] > \sum_{j=1}^{J}\sum_{t=0}^{\infty}\hat{w}_t e_t^j + \tilde{\Pi}^j \tag{1}$$

In a homework, you are asked to show that

$$\begin{aligned}\sum_{j=1}^{J}\tilde{\Pi}^j = &\sum_{t=0}^{\infty}\sum_{i_c=1}^{I_c}\hat{p}_{c,t}\tilde{y}_t^{i_c} - \hat{w}_t\tilde{n}_t^{i_c} - \hat{r}_t\tilde{k}_t^{i_c} \\ &+ \sum_{t=0}^{\infty}\sum_{i_c=1}^{I_c}\hat{p}_{x,t}\tilde{y}_t^{i_k} - \hat{w}_t\tilde{n}_t^{i_k} - \hat{r}_t\tilde{k}_t^{i_k}\end{aligned} \tag{2}$$

Now these are a bunch of infinite sums and without knowing something about them we cannot easily rearrange them. However, because they are all finite, we can do the rearranging and write

$$\begin{aligned}\sum_{j=1}^{J}\sum_{t=0}^{\infty}\hat{w}_t e_t^j + \tilde{\Pi}^j = \sum_{t=0}^{\infty}\Bigg[&\hat{p}_{c,t}\sum_{i_c=1}^{I_c}\tilde{y_t}^{i_c} + \hat{p}_{x,t}\sum_{i_{k=1}}^{I_k}\tilde{y}_t^{i_k} \\ &-\hat{r}_t\left(\sum_{i_c=1}^{I_c}\tilde{k}_t^{i_c} + \sum_{i_k=1}^{I_k}\tilde{k}_t^{i_k}\right) - \hat{w}_t\left(\sum_{i_c=1}^{I_c}\tilde{n}_t^{i_c} + \sum_{i_k=1}^{I_k}\tilde{n}_t^{i_k} - \sum_{j=1}^{J}e_t^j\right)\Bigg]\end{aligned} \tag{3}$$

Using a similar logic, we can write the left hand side of (1):

$$\sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\sum_{j=1}^{J}\tilde{c}_t^j + \hat{p}_{x,t}\sum_{j=1}^{J}\tilde{x}_t^j + \hat{w}_t\sum_{j=1}^{J}\tilde{\ell}_t^j - \hat{r}_t\sum_{j=1}^{J}\tilde{k}_t^j\right] \tag{4}$$

We can use feasibility constraint as defined above and write (3) as

$$\sum_{j=1}^{J}\sum_{t=0}^{\infty}\hat{w}_t e_t^j + \tilde{\Pi}^j = \sum_{t=0}^{\infty}\left[\hat{p}_{c,t}\sum_{j=1}^{J}\tilde{c}_t^j + \hat{p}_{x,t}\sum_{j=1}^{J}\tilde{x}_t^j - \hat{r}_t\sum_{j=1}^{J}\tilde{k}_t^j + \hat{w}_t\sum_{j=1}^{J}\tilde{\ell}_t^j\right]$$

This is exactly the same as the left hand side (4). Thus we have an inequality that states the above expression must be strictly less than itself. This is a contradiction. □

- The idea behind the first welfare theorem is what we mentioned in class. If there is something that makes everyone better off. Then they must not be able to afford it and if no one can afford an allocation it must not be feasible.

- The assumption that market value of allocations must be finite is a binding assumption. In fact, in overlapping generations model which I believe you will learn from Steve Spear, it is not hard to come up with examples where it breaks down and the CE is not pareto optimal.

- The first welfare theorem has two main implications:

  - Substantive implication: First, letting markets do their work create an optimal allocation. It would be impossible to improve everyone's life! Of course it is possible that society as a whole likes other things. For example, if there are some people whose endowment of capital and time is very low, they are miserable in a CE. Now if society likes to provide a sufficient level of utility to these guys, they must deviate from CE. FWT is useful because it identifies these trade-offs. it would mean that if we want to keep these households happy, someone must pay for it - one job for the government would be to enforce this but markets can also provide this service through charities perhaps.
  - Second practical implication: In solving for allocations, we wont have to worry about prices and just focus on allocations. This makes life easy in that it allows us to have an easier job of solving for the fixed point problem of guessing certain prices and then find prices so that markets clear. Instead we focus on solving a Pareto problem as I describe below.

- Pareto problem: it can be shown that if preferences are strictly concave then finding a pareto optimal allocation is equivalent to solving the following problem

$$\max_{z \in \mathcal{Z}} \sum_{j=1}^{J} \alpha^j U^j \left(\{ c_t^j, \ell_t^j \} \right)$$

  where $\alpha^j > 0$ is the welfare weight of household j. It is kind of saying that the allocations in a CE are equivalent to those in a planned economy where a fictitious planner just tells people how much to consume and invest and work and the planner cares about them at rate α^j.

- To be more precise, an allocation $\hat{z} \in \mathcal{Z}$ is pareto optimal if and only if there exists positive welfare weights $\alpha^j \geq 0$ such that

$$\hat{z} \in \arg\max_{z \in \mathcal{Z}} \sum_{j=1}^{J} \alpha^j U^j \left(\{ c_t^j, \ell_t^j \} \right)$$

Proof. The proof that the solution of the above problem is pareto optimal is straightforward. To see this, consider a $\hat{z}$ that solves the above maximization problem and suppose that an alternative allocation,$\tilde{z}$, exists which makes everyone weakly better off and some types strictly better off. If

$\alpha^j > 0$ for all j, then obviously this cannot be since the objective in the above optimization is higher under $\tilde{z}$ than z. When some of the α^j's are zero, then it must be that $\tilde{z}$ and $\hat{z}$ give the same level of utility. Since U^j is strictly concave, a convex combination of z and $\tilde{z}$ would deliver a higher utility. Therefore, this is a contradiction.

Now suppose $\tilde{z}$ is a pareto optimal allocation. Define the following set

$$A = \left\{ \boldsymbol{u} = \left(u^1, \cdots, u^J\right) \middle| \exists z \in \mathcal{Z}, u^j \leq U^j\left(z\right) \right\}$$

where we have abused the notation a bit by defining $U^j\left(z\right)$ to be U^j evaluated at the sequence of consumption and leisure for household j. Since the utility function is concave, then the above set must be a convex set (why?). Define set B as follows

$$B = \left\{ \boldsymbol{u} = \left(u^1, \cdots, u^J\right) \middle| \forall j, u^j \geq U^j\left(\tilde{z}\right) \right\}$$

Since $\tilde{z}$ is pareto optimal, it must be that

$$A \cap B = \left\{ \left(U^1\left(\tilde{z}\right), U^2\left(\tilde{z}\right), \cdots, U^J\left(\tilde{z}\right)\right) \right\}$$

(why?). Since A and B are both convex sets in $\mathbb{R}^J$, we can apply the separating hyperplane theorem. As a result, vectors $\boldsymbol{\alpha} \in \mathbb{R}^J, \boldsymbol{\alpha} \neq 0$ and a constant c must exist such that

$$\langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle \geq c \text{ if } \boldsymbol{u} \in B$$
$$\langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle \leq c \text{ if } \boldsymbol{u} \in A$$

where $\langle \cdot, \cdot \rangle$ is the product in $\mathbb{R}^J$:

$$\langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle = \sum_{j=1}^{J} \alpha^j u^j$$

Note that since $\tilde{\boldsymbol{u}} = \left(U^1\left(\tilde{z}\right), U^2\left(\tilde{z}\right), \cdots, U^J\left(\tilde{z}\right)\right)$ belongs to both sets, we must have

$$c = \langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} \rangle$$

This together with the statement of the separating hyperplane theorem implies that

$$\langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle \leq \langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} \rangle, \forall \boldsymbol{u} \in A$$

and by definition of A

$$\sum_j \alpha^j U^j\left(z\right) \leq \sum_j \alpha^j U^j\left(\tilde{z}\right), \forall z \in \mathcal{Z}$$

which is equivalent to

$$\tilde{z} \in \arg\max_{z \in \mathcal{Z}} \sum \alpha^j U^j\left(z\right)$$

The only thing that remains to be shown is that $\alpha^j \geq 0$. To show this, let $i \in \{1, \cdots, J\}$ and

$$\boldsymbol{e}^i = \left(\underbrace{0, \cdots, 0}_{i-1 \text{ times}}, 1, 0, \cdots, 0 \right)$$

Then since $\tilde{\boldsymbol{u}} + \boldsymbol{e}^i \in B$, we must have that

$$c = \langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} \rangle \leq \left\langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} + \boldsymbol{e}^i \right\rangle = \langle \boldsymbol{\alpha}, \tilde{\boldsymbol{u}} \rangle + \alpha^i \Rightarrow 0 \leq \alpha^i$$

which completes the proof. □

- **Aggregation of households:** Note that when utilities are time-separable and people have the same discount rate, the objective in the above optimization can be written as

$$\sum_{t=0}^{\infty} \beta^t \sum_{j=1}^{J} \alpha^j u^j \left(c_t^j, \ell_t^j\right)$$

This is basically saying that the economy we described is equivalent to an economy where a single representative household is making decisions about how much to consume, work and invest. This even further simplifies our life in solving this problem. Note that it does not necessarily imply that distributions do not matter. In particular, we can define the following aggregate utility function

$$U\left(c, \ell; \alpha\right) = \max \sum_{j=1}^{J} \alpha^j u^j \left(c^j, \ell^j\right)$$

subject to

$$\sum_{j=1}^{J} c^j = c, \sum_{j=1}^{J} \ell^j = \ell$$

Then the planning problem is simply given by

$$\max_{c_t, \ell_t, k_t^{i_c}, n_t^{i_c}, k_t^{i_k}, n_t^{i_k}} \sum_{t=0}^{\infty} \beta^t U\left(c_t, \ell_t; \alpha\right) \tag{P}$$

subject to

$$\begin{aligned} c_t &= \sum_{i_c=1}^{I_c} F^{i_c}\left(k_t^{i_c}, n_t^{i_c}\right) \\ x_t &= \sum_{i_k=1}^{I_k} F^{i_k}\left(k_t^{i_k}, n_t^{i_k}\right) \\ k_{t+1} &= \left(1-\delta\right) k_t + x_t \\ \bar{e} - \ell &= \sum_{i_c=1}^{I_c} n_t^{i_c} + \sum_{i_k=1}^{I_k} n_t^{i_k} \\ k_t &= \sum_{i_c=1}^{I_c} k_t^{i_c} + \sum_{i_k=1}^{I_k} k_t^{i_k} \end{aligned}$$

As you can seen this problem at least gets rid of a whole bunch of heterogeneity inherent in our initial problem. Now there is a class of utility functions that makes life even simpler. Suppose, for example that

$$u^j\left(c, \ell\right) = \frac{\left(c^\theta \ell^{1-\theta}\right)^{1-\sigma}}{1-\sigma}, \forall j$$

Then we have

$$U(c,\ell;\alpha) = \max \sum_{j=1}^{J} \alpha^j \frac{\left(c_j^\theta \ell_j^{1-\theta}\right)^{1-\sigma}}{1-\sigma}$$

subject to

$$\sum_{j=1}^{J} c_j = c, \sum_{j=1}^{J} \ell_j = \ell$$

Taking first order conditions from this problem we have

$$\alpha^j \theta \frac{\left(c_j^\theta \ell_j^{1-\theta}\right)^{1-\sigma}}{c_j} = \lambda_c, \alpha^j (1-\theta) \frac{\left(c_j^\theta \ell_j^{1-\theta}\right)^{1-\sigma}}{\ell_j} = \lambda_l$$

Let's do some algebra now. If we divide the two equations, we get

$$\frac{\ell_j}{c_j} = \frac{1-\theta}{\theta}\frac{\lambda_c}{\lambda_l}, \forall j = 1, \cdots, J$$

This means that

$$\frac{\ell_j}{c_j} = \frac{\ell}{c} = \frac{1-\theta}{\theta}\frac{\lambda_c}{\lambda_l}$$

Now if define this ratio to be some constant κ, then we have

$$\alpha^j \theta c_j^{-\sigma} \kappa^{1-\sigma} = \lambda_c \to \frac{c_j}{c_i} = \left(\frac{\alpha^j}{\alpha^i}\right)^{\frac{1}{\sigma}} \to c_i = \frac{(\alpha^i)^{\frac{1}{\sigma}}}{\sum_{j=1}^{J} (\alpha^j)^{\frac{1}{\sigma}}} c$$

Similarly

$$\ell_i = \frac{(\alpha^i)^{\frac{1}{\sigma}}}{\sum_{j=1}^{J} (\alpha^j)^{\frac{1}{\sigma}}} \ell$$

Thus the utility is given by

$$\begin{aligned} U(c,\ell;\alpha) &= \sum_{j=1}^{J} \alpha^j \frac{(\alpha^j)^{\frac{1-\sigma}{\sigma}} \left(c^\theta \ell^{1-\theta}\right)^{1-\sigma}}{(1-\sigma)\left(\sum_{j=1}^{J} (\alpha^j)^{\frac{1}{\sigma}}\right)^{1-\sigma}} \\ &= \left[\sum_{j=1}^{J} (\alpha^j)^{\frac{1}{\sigma}}\right]^{\sigma} \frac{\left(c^\theta \ell^{1-\theta}\right)^{1-\sigma}}{1-\sigma} \end{aligned}$$

This implies that we can write the objective function in the planning problem P, as

$$\left[\sum_{j=1}^{J} (\alpha^j)^{\frac{1}{\sigma}}\right]^{\sigma} \sum_{t=0}^{\infty} \beta^t \frac{\left(c_t^\theta \ell_t^{1-\theta}\right)^{1-\sigma}}{1-\sigma}$$

This means that distribution of allocations actually do not affect aggregate outcomes since α's have no effect on the optimal decision. So we can just separate the problem of determining optimal allocations from that of their distribution among households.

- This class of utility function is a special case of a more general class of utility functions that satisfy the so-called Gorman aggregation. One can show that this is the case if and only if preferences are homothetic and identical, i.e.,

$$(c, \ell) \sim_j (c', \ell') \Leftrightarrow (\lambda c, \lambda \ell) \sim_j (\lambda c', \lambda \ell'), \forall \lambda > 0$$

- Note that in general the problem is still quite tractable because conditional on the vector α, we can characterize - as we will see - the evolution of aggregate allocations. However, in case of general preferences that are non-homothetic, the distribution of resources interacts with aggregate behavior of the economy. See the paper by Chatterjee (1994).

- A simplification that we have made in our time-separable preferences above was the assumption of constant discounting. This assumption allows us to write the problem recursively in an easy fashion - as we will show. It, however, implies that the rate at which people discount the future is the same independent of where they stand in time. In other word, at $t = 0$, the value a unit of utility at time 6 relative to time 5 is β. Now if the individual goes to period $t = 1$, the same is true. Now suppose instead utilities evaluated at $t = 0$ are given by

$$\sum_{t=0}^{\infty} \beta_t u_t$$

  where β_t is not an exponential series. Then we have to specify preferences for an individual who is making decisions at $t = 1$. The bottomline is we cannot come up with a utility function where the two individuals agree on the way they evaluate future utilities. This creates what is known as a time-inconsistency problem in decision making where planning made by an individual at $t = 0$ is not consistent with optimal decisions of that same individual at $t = 1$. You can read more about this in the seminal paper by Laibson (1997).

- **Second Welfare theorem**: While we have talked about pareto optimality of CE, we have not really explained how should one go back from a general P.O. allocation to a C.E. (basically how to construct prices). It turns out to do this generally, we would need transfers. In particular, for a sequence of transfers $\{T^j\}_{j=1}^J$ where $\sum_{j=1}^J T^j = 0$, we modify the budget constraint in the definition of C.E. to

$$\sum_{t=0}^{\infty} \left(p_{c,t} c_t^j + p_{x,t} x_t^j\right) \leq \sum_{j=0}^{\infty} \left[w_t \left(e_t^j - \ell_t^j\right) + r_t k_t^j\right] + \Pi^j + T^j$$

A *C.E. with transfers* given transfers T^j can be defined accordingly using the above modified budget constraint. We thus have the following result, also known as the *Second Welfare Theorem*:

Theorem 2. *Consider a P.O. allocation, $\hat{z} \in \mathcal{Z}$. Then there exists transfers $\left\{\hat{T}^j\right\}_{j=1}^J$ together with a vector of prices $\{\hat{p}_{c,t}, \hat{p}_{x,t}, \hat{w}_t, \hat{r}_t\}$ so that they together with $\hat{z}$ constitute a C.E. with transfers.*

Proof. In order to prove existence of prices, we would need to resort to a separating hyperplane theorem for infinite dimensional spaces which is really not that economically insightful and beyond our scope here - but mathematical very interesting and challenging. You can read the proof of the finite dimensional case in standard microeconomics textbooks such as the one by Mas-colell, Whinston and Green. More general proofs can be found in various papers by Zame, Mas-colell and Bewley. □

- This theorem basically states that we can achieve any pareto allocation we want as long as we have individual specific transfers. Unfortunately, this is too strong of an assumption. In other words, a government or planner, needs to know various details about households' wealth and productivity and other sources of endowment. This is perhaps too costly for the government - almost impossible to verify someone's productivity in the labor market even if we can observe out their wealth. Once it is hard for the government to observe people's characteristics, the second welfare theorem disappears and we do not get to implement any pareto optimal allocation that we want. In fact, this potentially creates a trade-off between the desired allocation of goods and services across people and economic efficiency. This is one of the main starting points for the public finance literature that studies redistributive taxation and its interactions with efficiency. We will come back to this later during the course.

- **Aggregation of production functions:** Standard results in micro allows us to always aggregate production function, again you can read about this in micro textbooks. This implies that we can basically represent the production side of the economy with two production functions:

$$F^c\left(k^c, n^c\right), F^k\left(k^k, n^k\right)$$

- Another assumption that makes our life really simple is to assume that the two production functions above are identical and are both constant returns to scale and are strictly concave. In this case, economic profits Π^j will always be zero in equilibrium and prices of consumption and investment will be equal. We can thus aggregate the above two production functions into just one production function $F\left(k, n\right)$ and write feasibility as

$$c_t + x_t = y_t = F\left(k_t, n_t\right)$$

  This is the familiar NIPA equation from intermediate macro - since this is a closed economy without a government, NX and G do not show up here!

- *Question: what is the NIPA equation look like in case of the disaggregated economy?*

- Now that we have significantly simplified this problem, we are ready to try to solve to do this let us restate the problem again

$$\max \sum_{t=0} \beta^t U\left(c_t, \ell_t\right)$$

subject to

$$\begin{aligned} c_t + x_t &= F(k_t, n_t) \\ k_{t+1} &= (1-\delta) k_t + x_t \\ \bar{e} - n_t &= \ell_t \\ k_0 &: \text{given} \\ x_t, c_t, k_t, n_t, \ell_t &\geq 0 \end{aligned}$$

where we have imposed that aggregate leisure in the economy is constant. One thing that is important to note is that the above programming problem only depends on k_0 - the initial state of the economy. As we will see this makes the analysis tractable - dealing with one state variable is really easy!

- In order to use tools from Dynamic Programming developed by Richard Bellman and presented also in Stokey and Lucas Jr (1989), we need one more step. We define the following auxiliary utility function

  $$\hat{U}(k, k') = \max_{c,\ell,n\geq 0} U(c, \ell)$$

  subject to

  $$\begin{aligned} c &= F(k, n) + (1-\delta) k - k' \\ n &= \bar{e} - \ell \end{aligned}$$

  Note that if we assume that the solution of the above problem is interior, then taking first order conditions and combining them, we will have

  $$\frac{U_\ell(c, \ell)}{U_c(c, \ell)} = F_n(k, \bar{e} - \ell)$$

  This equation is quite intuitive. The right hand side is the marginal benefit of one more hours of work. The left hand side is its marginal cost. U_ℓ is the cost of giving up one unit of leisure and by dividing it by U_c we change the units of this cost from being in utility terms to being in consumption terms. We will also refer to the solution of this programming problem as $\tilde{n}(k, k'), \tilde{c}(k, k')$. A special case of this is one where there is labor supply is inelastic, i.e., households dont care about leisure, $u_\ell^j(c, \ell) = 0$ in which case $\ell = 0$ or $n = \bar{e}$. In this case, $\hat{U}(k, k') = U(F(k, \bar{e}) + (1-\delta) k - k', 0) = u(F(k, \bar{e}) + (1-\delta) k - k')$ - with a slight abuse of notation. For now, let's focus on this case.

- Given this auxiliary utility function, the above programming problem is equivalent to

  $$\max_{k_t} \sum_{t=0}^{\infty} \beta^t u(F(k_t, \bar{e}) + (1-\delta) k_t - k_{t+1})$$

  subject to

  $$\begin{aligned} (1-\delta) k_t \leq k_{t+1} &\leq F(k_t, \bar{e}) + (1-\delta) k_t \\ k_0 &: \text{ given} \end{aligned}$$

  Question: where are the inequalities coming from?

- **Dynamic Programming:** If we refer to the value of the above optimization problem as $V(k_0)$, then this value function must solve the following functional equation

$$V(k) = \max_{k' \in \Gamma(k)} u(F(k, \overline{e}) + (1-\delta)k - k') + \beta V(k')$$

where $\Gamma(k) = [(1-\delta)k, (1-\delta)k + F(k, \overline{e})]$.

- Standard results - by the end of the semester anyways! - show that
  - if U and F are continuous, then $V(\cdot)$ is unique. The idea is that if we define the following transformation on the space of continuous and bounded functions

$$Tv(k) = \max_{k' \in \Gamma(k)} u(F(k, \overline{e}) + (1-\delta)k - k') + \beta v(k')$$

    for an arbitrary function $v(\cdot)$, then V must be a fixed point of this transformation. The proof basically goes by showing that T is a contraction mapping and therefore has a unique fixed-point which would be V.
  - If U is strictly concave and strictly increasing, F is strictly concave and strictly increasing, then V is strictly concave, and strictly increasing.
  - If U and F are continuously differentiable, then V is also continuously differentiable.
  - For any continuous function, $V_0(k)$, we have

$$\lim_{n\to\infty} T^n V_0(k) = V(k)$$

    where $T^n V_0(k)$ is the result of applying T to V_0, n-times. Also, the convergence occurs according to the sup-norm on the space of functions. This is very useful result for computations.

- If the solution for k' in the optimization problem above is interior, we can write

$$-u'(F(k, \overline{e}) + (1-\delta)k - k') + \beta V'(k') = 0 \tag{5}$$

We refer to the solution of the optimization by $\tilde{k}(k)$, what is also referred to as a *policy function*. Note that we can calculate the derivative of the value function, $V(k)$ by applying the Envelope theorem to the maximization above which implies that

$$V'(k) = u'\left(F(k, \overline{e}) + (1-\delta)k - \tilde{k}(k)\right) \cdot (F_k(k, \overline{e}) + 1 - \delta)$$

We thus have

$$V'\left(\tilde{k}(k)\right) = u'\left(F\left(\tilde{k}(k), \overline{e}\right) + (1-\delta)\tilde{k}(k) - \tilde{k}\left(\tilde{k}(k)\right)\right) \cdot \left(F_k\left(\tilde{k}(k), \overline{e}\right) + 1 - \delta\right)$$

If we replace this in (5) we get

$$u'\left(F(k, \overline{e}) + (1-\delta)k - \tilde{k}(k)\right) = \beta u'\left(F\left(\tilde{k}(k), \overline{e}\right) + (1-\delta)\tilde{k}(k) - \tilde{k}\left(\tilde{k}(k)\right)\right) \cdot \left(F_k\left(\tilde{k}(k), \overline{e}\right) + 1 -\right.$$

We can rewrite this as

$$u'(c_t) = \beta u'(c_{t+1}) [1 - \delta + F_k(k_{t+1}, \overline{e})]$$

This equation is called an Euler equation and intuitively captures the trade-offs in optimal investment decision. When the consumer at date t gives up ε units of consumption (where ε is small) and invests it, capital stock in period $t+1$ goes up by ε. If we assume that k_{t+2} remains, i.e., period $t+2$ onwards, remains unchanged, then the resources available to the consumer in period $t+1$ are given by $(F_k(k_{t+1}, \overline{e}) + 1 - \delta)\varepsilon$. This is because the extra output produced with the extra units of capital plus the undepreciated value of capital are all available for consumption. The marginal cost (in terms of utils) of cutting consumption at t by ε is $\beta^t u'(c_t)\varepsilon$ while the marginal benefit of increasing consumption at $t+1$ is $\beta^{t+1}u'(c_{t+1}) \times (1 - \delta + F_k(k_{t+1}, \overline{e}))\varepsilon$. Now at the optimum and for a small value for ε, marginal benefit must be equal to the marginal cost and thus we get the Euler equation.

- **Evolution of capital over time:** In order to understand the evolution of capital over time, we need basic characterization of the policy function $\tilde{k}(k)$. We can use the recursive formulation to show that the policy function, $\tilde{k}(k)$, is increasing in k. We can further show that there are two value of k that satisfy $\tilde{k}(k) = k$. One of them is 0 and the other one is some positive value $k^* = \tilde{k}(k^*)$. It can then be shown that if $k_0 > 0$, then

$$\lim_{n\to\infty} \tilde{k}^n(k_0) = k^*$$

Now this result is pretty important, because it states that capital accumulation on its own cannot generate long-term growth. In other words, the fact that we have seen persistent growth in the past 200 years through the lens of this model can only be explained if we were on a very long and slow transition path to the steady states. There are various reasons to believe that this cannot be true. We will next discuss various implications of this growth model.

  - In words, the only we can have growth in this model is by accumulation of capital. For high enough values of capital, decreasing returns to scale kick in and do not let the economy grow
  - To summarize, this model - in Steady State - explains some of the Kaldor facts, namely that factor shares are stable and that capital labor is stable but it cannot generate growth in GDP per capita - aside from transition - and in real wages. To calculate factor prices which with a little abuse of notation we call w_t and r_t , we must solve the firm's problem

$$\max_{k,n} F(k, n) - w_t n - r_t k$$

We therefore have

$$F_n(k_t, \overline{e}) = w_t$$
$$F_k(k_t, \overline{e}) = r_t$$

where $k_t = \tilde{k}^t (k_0)$. This implies that in the steady state, real wages and real interest rates are constant - the second one is a Kaldor facts. Over the course of transition, real wages rise and real interest rates fall. Moreover, factor shares are given by

$$\alpha_{k,t} = \frac{r_t k_t}{F(k_t, \overline{e})} = \frac{F_k(k_t, \overline{e}) k_t}{F(k_t, \overline{e})}$$
$$\alpha_{n,t} = \frac{w_t \overline{e}}{F(k_t, \overline{e})} = \frac{F_n(k_t, \overline{e}) \overline{e}}{F(k_t, \overline{e})}$$

Since F is constant returns to scale, factor shares sum up to 1. Now, what they look like in the steady state and over transition depends on the shape of the production function.

* Cobb-Douglass: $F(k,n) = Ak^\alpha n^{1-\alpha}$, then

$$\alpha_{k,t} = \frac{\alpha A k_t^{\alpha-1} \overline{e}^{1-\alpha} k_t}{A k_t^\alpha \overline{e}^{1-\alpha}} = \alpha, \alpha_{n,t} = 1-\alpha$$

* CES: $F(k,n) = A[\alpha k^\gamma + (1-\alpha) n^\gamma]^{\frac{1}{\gamma}}$, then

$$F_k = A\alpha k^{\gamma-1} [\alpha k^\gamma + (1-\alpha) n^\gamma]^{\frac{1}{\gamma}-1}$$
$$\alpha_{k,t} = \frac{A\alpha k_t^{\gamma-1} [\alpha k_t^\gamma + (1-\alpha) \overline{e}^\gamma]^{\frac{1}{\gamma}-1} k_t}{A [\alpha k_t^\gamma + (1-\alpha) \overline{e}^\gamma]^{\frac{1}{\gamma}}}$$
$$= \frac{\alpha k_t^\gamma}{\alpha k_t^\gamma + (1-\alpha) \overline{e}^\gamma}$$
$$\alpha_{n,t} = \frac{(1-\alpha) \overline{e}^\gamma}{\alpha k_t^\gamma + (1-\alpha) \overline{e}^\gamma}$$

Under this specification, capital share increases during transition while labor share falls during transition. Given that factor shares seem to be stable even if we believe this model, Cobb-Douglas seems like a better choice at least for long-term analysis. It is true that labor shares - corporate labor share - have declined in some countries in the past twenty years. For more on this see the paper by Karabarbounis and Neiman (2014).

- A very common assumption: $F(K,L) = AK^\alpha L^{1-\alpha}$.
- Lucas' observation: India's GDP per capita is 1/15 times US's. What does this imply about interest rate differences? In particular, suppose we have

$$Y_{US} = AK_{US}^\alpha L_{US}^{1-\alpha}, Y_{India} = AK_{India}^\alpha L_{India}^{1-\alpha}$$
$$y_{US} = Ak_{US}^\alpha, y_{India} = Ak_{India}^\alpha$$

where y_i is GDP per capita and k_i is capital per capita. If this model is true, then

$$\frac{r_{US}}{r_{India}} = \left(\frac{y_{US}}{y_{India}}\right)^{\frac{\alpha-1}{\alpha}} = (15)^{-2} = 1/225$$

where we have assumed $\alpha = 1/3$. If this is true, then capital must be flowing from rich to poor countries where as in reality it does not. Why?

  - Productivity differences due to human capital
  - productivity differences due to other factors: history, language, technology
  - Taxes and other sources of government expropriation: the return is really not r_{India} as defined above but this is the before taxes/expropriation rental rate of capital
  - Maybe production functions are different: have different factor shares

- Gollin's paper: labor shares are mismeasured - not very different across countries once we account for self-employed.

- Productivity differences: human capital vs other things - evidence by Hall and Jones (1999). They assume

$$Y_i = K_i^{\alpha} \left(A_i H_i\right)^{1-\alpha}$$
$$H_i = e^{\phi(E_i)} L_i$$

  where E_i is schooling, L_i is total number of workers, and H_i is human capital. As we mentioned in class, the reason we use this specification is because we can connect it to the so-called Mincer regression that people run for the effect of schooling on wages. In particular, real wages in this model are given by

$$w_i = F_L = (1-\alpha) K_i^{\alpha} A_i^{1-\alpha} e^{(1-\alpha)\phi(E_i)} L_i^{-\alpha}$$

  taking logs we get

$$\log w_i = \text{stuff } + (1-\alpha)\phi(E_i)$$

  So we can run a regression of wages on years of schooling within a country and have an estimate ϕ (we can also do cross-country regression which seems more feasible as we dont need micro-data for each country)

- So we have

$$y_i = \left(\frac{K_i}{Y_i}\right)^{\frac{\alpha}{1-\alpha}} A_i h_i$$

  where h_i is human capital per capita. Notice that we have also put capital output ratio since as they argue is not susceptible to certain biases. As the figure in the slides show, doing this calculation and basically running a regression of productivity on GDP per capita gives us a coefficient of 0.6 and R-squared of 0.79. So most of the variation of GDP per capita across countries is coming from variations in productivity.

- One test of the model is on how fast is it that countries are getting closer to each other. Note that the model says that similar countries with similar productivities, discount factors and utility functions should converge to the same steady state.

- Can learn something about whether the model is correct from how fast countries converge to each other. Barro and Sala-i Martin (1992) (Cross-state regressions: 2-3% per year convergence to steady state)

$$\log\left(y_{i,t}/y_{i,t-1}\right) = \text{stuff} - \left(1-e^{-\beta}\right)\log y_{i,t-1}$$

Where is this coming from? log-linearization of the model. We will come back to this later. They show$\beta \approx 0.02$. States gets closer to each other 2% per year. Need a share of capital around 0.8!!! Is this too high? Maybe but maybe not. In particular, suppose that there are two types of capital, physical and human capital:

$$Y = AK^{\alpha}H^{\beta}L^{1-\alpha-\beta}$$

If capital and human capital have the same accumulation equation, i.e., the same depreciation rate, then I leave it to you to show that we can combine the two capitals into one hybrid capital. This capital's share in GDP is given by $\alpha + \beta$. Then a share of 0.8 is not that too far fetched.

- **Population Growth:** So given that the model does not have growth and we dont have a lot of evidence that say growth has been declining over the years, we need to create growth in some other fashion. One natural source of growth that we ignored is growth in the labor force or population. So suppose that

  $$N_t = N_0 (1+n)^t$$

  and let us normalize total leisure to 1. Then feasibility constraint is given by

  $$C_t + K_{t+1} = (1-\delta) K_t + F(K_t, N_t)$$

  Notice that I have switched to capital letters to describe aggregate allocations.

  - When there is population growth it is always tricky to write down a social welfare function. The problem is that there are new people being born every period and how do we construct social welfare function. To see this, let us index the set of people born at t by $\mathcal{I}_t$ which is some interval of length, nN_{t-1}. For simplicity we can write $\mathcal{I}_t = [N_{t-1}, N_t]$. Then the utility of an individual $i \in \mathcal{I}_t$ that is born at t, is given by

    $$U^i = \sum_{s=t}^{\infty} \beta^{s-t} u\left(c_s^i\right)$$

    where c_s^i is the per capita consumption of this individual. Now a general social welfare function as

    $$\sum_{t=0}^{\infty} \int_{\mathcal{I}_t} g_t(i)\, U^i di$$

    where $g_t(i)$ is the welfare weight on person i at t. Now suppose we impose that everyone's consumption is the same. Then

    $$\sum_{t=0}^{\infty} \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \int_{\mathcal{I}} g_t(i)\, di$$

    where c_t is per capita consumption at t. Let's call $\hat{g}_t = \int_{\mathcal{I}_t} g_t(i)\, di$.
    Changing the order of summation the objective function becomes

    $$\sum_{t=0}^{\infty} u(c_t) \sum_{s \le t} \beta^{t-s} \hat{g}_s$$

Suppose we let

$$g_t(i) = \beta^t \to \hat{g}_t = \beta^t (N_t - N_{t-1})$$

where we assume $N_{-1} = 0$. Then

$$\begin{aligned}\sum_{s=0}^{t} \beta^{t-s} \hat{g}_s &= \sum_{s=0}^{t} \beta^{t-s} \beta^s (N_s - N_{s-1}) \\ &= \beta^t N_t\end{aligned}$$

so then the social welfare function becomes

$$\sum_{t=0}^{\infty} \beta^t N_t u(c_t)$$

We can also consider other social welfare functions: for example one that only puts weight on the initial generation:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Question: *If you were to solve the C.E. of this economy – with time-0 trading which is a bit weird! – what would be the welfare weights that rationalize it?*
- We can also consider other social welfare functions: for example one that only puts weight on the initial generation:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

This is what we discussed in class.

- Note that sometimes this can lead to time inconsistency. For example suppose that

$$g_t(i) = \beta^t \frac{1}{N_t - N_{t-1}} \to \sum_{s \le t} \beta^{t-s} \hat{g}_s = (t+1) \beta^t$$

which is not time consistent. See the paper by Jackson and Yariv (2015).

- Let us choose the first example of the social welfare function. Then the problem of maximizing utility becomes

$$\max \sum_{t=0}^{\infty} \beta^t N_t u\left(\frac{C_t}{N_t}\right)$$

subject to

$$C_t + K_{t+1} = (1-\delta) K_t + F(K_t, N_t)$$

- Let's first get a couple of technical issues out of the way. Note that potentially things can grow without bounds. So we can quite apply the dynamic programming techniques here since our proof of principle of optimality relied on that. Second, if population growth is really fast, then we might run into trouble since the sum in the objective could be infinity. We can deal with both of these issues. As for the first one, we switch back to small letters. In other words, let

$$k_t = \frac{K_t}{N_t}, c_t = \frac{C_t}{N_t}$$

Then dividing the feasibility constraint by N_t and using CRS assumption, we have

$$\begin{aligned} c_t + \frac{K_{t+1}}{N_t} &= (1-\delta)\, k_t + F(k_t, 1) \\ c_t + \frac{N_{t+1}}{N_t}\frac{K_{t+1}}{N_{t+1}} &= (1-\delta)\, k_t + F(k_t, 1) \\ c_t + (1+n)\, k_{t+1} &= (1-\delta)\, k_t + F(k_t, 1) \end{aligned}$$

This is very similar to what we had before so presumably because of decreasing returns to scale with respect to capital we can apply the same dynamic programming techniques as before.
Now given the this modification, the objective is given by

$$N_0 \sum_{t=0}^{\infty} (\beta (1+n))^t\, u(c_t)$$

Thus, this is like a problem where the discount rate in the planning problem is $\beta(1+n)$ and impose the assumption that $\beta(1+n) < 1$. The functional equation version of it is

$$V(k) = \max_{k'} u((1-\delta)\, k + F(k,1) - (1+n)\, k) + \beta(1+n)\, V(k')$$

This problem is not very different than the old problem. In particular, its Euler equation is given by

$$(1+n)\, u_{ct} = \beta(1+n)\left[F_k(k_{t+1}, 1) + 1 - \delta\right] u_{ct+1}$$

which is identical to the one we had before except in terms of per-capita stuff - capital and consumption.

- How are things look like in the long-run? They look like something that we referred to as balanced growth path, all per capita variables growing at the same rate. Note that since we have decreasing returns to scale with respect to capital-per-capita, again the capital-per-capita converges to a level k^* which is given by

$$\beta\left[F_k(k^*, 1) + 1 - \delta\right] = 1 \tag{6}$$

All of this implies that growth of per capita variables in the BGP - balanced growth path - is 0. So again we can generate long growth.

- **Productivity Growth.** The last place to put growth in is productivity. It turns out there are many ways to do this:

- **Labor Augmenting Productivity Growth or Harrod-Neutral:**

$$Y_t = F\left(K_t, A_t L_t\right)$$

where $A_t = A_0\left(1+g\right)^t$ and $N_t = \left(1+n\right)^t N_0$. A balanced growth path for this economy is one where all per capita variables grow at rate $\hat{g}$. We must have

$$\left(1+\hat{g}\right)^t\left(1+n\right)^t Y_0 = F\left(K_0\left(1+\hat{g}\right)^t\left(1+n\right)^t, \left(1+g\right)^t A_0 N_0\left(1+n\right)^t\right)$$

Thus the economy can potentially grow in the long run. The question is how can we show that.

  * We are going to assume that

$$u\left(c\right) = \begin{cases} \frac{c^{1-\sigma}}{1-\sigma} & \sigma \neq 1 \\ \log c & \sigma = 1 \end{cases}$$

We also define the following variable

$$X_t = A_t N_t$$

We also define

$$\hat{c}_t = \frac{C_t}{X_t}, \hat{k}_t = \frac{K_t}{X_t}$$

Then the feasibility constraint becomes

$$\hat{c}_t + \left(1+n\right)\left(1+g\right)\hat{k}_{t+1} = \hat{k}_t\left(1-\delta\right) + F\left(\hat{k}_t, 1\right)$$

We can also write the objective as

$$\sum_{t=0}^{\infty}\left(\beta\left(1+n\right)\left(1+\hat{g}\right)^{1-\sigma}\right)^t u\left(\hat{c}_t\right)$$

Then the Bellman equation associated with this is

$$V\left(\hat{k}\right) = \max_{\hat{k}'} u\left(\left(1-\delta\right)\hat{k} + F\left(\hat{k}, 1\right) - \left(1+n\right)\left(1+\hat{g}\right)\hat{k}\right) + \beta\left(1+n\right)\left(1+\hat{g}\right)^{1-\sigma} V\left(\hat{k}'\right)$$

The Euler equation is given by

$$u'\left(\hat{c}_t\right) = \beta\left(1+g\right)^{-\sigma}\left[1-\delta + F_{k,t+1}\left(\hat{k}_{t+1}, 1\right)\right] u'\left(\hat{c}_{t+1}\right)$$

As before we can show that the optimal allocations in this economy converge:

$$\lim_{t\to\infty}\hat{k}_t = \hat{k}^*$$

where

$$1 = \beta\left(1+g\right)^{-\sigma}\left[1-\delta + F_k\left(\hat{k}^*, 1\right)\right]$$

* Thus a unique balanced growth path exists and the economy always converges to that. The growth rate is approximately given by$g$. Now, let's see what happens to other Kaldor facts, we have

$$
\begin{aligned}
w_t &= A_t F_N\left(K_t, A_t N_t\right) \text{ firms' first order condition with } w_t \text{ :the real wage} \\
&= A_t F_N\left(\frac{K_t}{A_t N_t}, 1\right) \text{ since } F \text{ is homogenous of degree 1 - show this!} \\
&= A_t F_N\left(\hat{k}^*, 1\right)
\end{aligned}
$$

Note that the above holds on a balanced growth path. So wages also grow at a constant rate.

$$
\begin{aligned}
r_t &= F_K\left(K_t, A_t N_t\right) = F_K\left(\frac{K_t}{A_t N_t}, 1\right) \\
&= F_K\left(\hat{k}^*, 1\right)
\end{aligned}
$$

which is constant. Note that BGP level of capital-output ratio in this economy is given by

$$
\begin{aligned}
\frac{K_t}{Y_t} &= \frac{\hat{k}^* A_t N_t}{F\left(K_t, A_t N_t\right)} = \frac{\hat{k}^*}{F\left(\frac{K_t}{A_t N_t}, 1\right)} \\
&= \frac{\hat{k}^*}{F\left(\hat{k}^*, 1\right)}
\end{aligned}
$$

In the models without long-run growth capital -output ratio is given by

$$
\frac{k^*}{F\left(k^*, 1\right)}
$$

where k^* satisfies (6). Note that when $g > 0$, then $k^* > \hat{k}^*$ and as a result, capital-output ratio is lower in the economy with growth. Intuitively, since in this economy consumption is growing people will have a lower marginal utility of consumption tomorrow relative to today. As a result, the rate of return on capital should increase which means an increase in capital-output ratio. Therefore, the steady state level of capital-output ratio depends on the growth rate of the economy.

* This is basically our first model that can match the Kaldor facts and growth is all determined by productivity growth. It is somewhat disappointing however as growth does not depend on anything like taxes/property rights; economic decision makings, etc. Is this a bad result? Perhaps. It is because we see a lot of countries where property rights are not enforced and governments expropriate stuff and they are not growing very fast. *Question:* Can the causality be reversed?

    * Note also that we effectively have increaisng returns to scale here. To see that, we have to basically think about productivity as another input into production and then you realize that when all inputs go up by the same factor, output goes up by more than that. This idea shows up in other models of growth namely those of endogenous growth – although it is not necessary as we see shortly.

  – **Hicks neutral productivity growth**:

$$Y_t = A_t F(K_t, N_t)$$

  For Cobb-Douglass – $F(K,N) = K^{\alpha} N^{1-\alpha}$ – this is equivalent to labor-augmenting productivity growth while the BGP growth rate has to be adjusted. In particular, we can show that the growth rate of the BGP in this case is $(1+g)^{\frac{1}{1-\alpha}} - 1$ – *Show this!* Therefore the long-run growth rate of the economy depends on the share of labor and capital in the production function.

  – **Capital Augmenting Productivity Growth:**

$$Y_t = F(A_t K_t, L_t)$$

  again with Cobb-douglas it is similar to what we have before. We also have a stronger result due to Uzawa (1961):

  – Uzawa's theorem - much stronger statements are also true (see paper by Grossman et al. (2016)):

  – Suppose that $Y = F(A_t K, L)$ where $\frac{A_{t+1}}{A_t} = 1+g$, then the only way to have balanced growth is to have $F(K,L) = \hat{A} K^{\alpha} L^{1-\alpha}$.

  – This is a problem: How can we think of falling labor share, changes in investment good prices, etc?

References

Barro, R. J. and X. Sala-i Martin (1992): "Convergence," *Journal of Political Economy*, 223–251. 16

Burns, A. F. and W. C. Mitchell (1946): "Measuring business cycles," *NBER Books*. 1

Chatterjee, S. (1994): "Transitional dynamics and the distribution of wealth in a neoclassical growth model," *Journal of Public Economics*, 54, 97–119. 10

Grossman, G. M., E. Helpman, E. Oberfield, and T. Sampson (2016): "Balanced growth despite Uzawa," Tech. rep., National Bureau of Economic Research. 22

Hall, R. E. and C. I. Jones (1999): "Why Do Some Countries Produce So Much More Output Per Worker Than Others?" *The Quarterly Journal of Economics*, 114, 83–116. 16

Jackson, M. O. and L. Yariv (2015): "Collective Dynamic Choice: The Necessity of Time Inconsistency," *American Economic Journal: Microeconomics*, 7, 150–178. 18

Karabarbounis, L. and B. Neiman (2014): "The Global Decline of the Labor Share*." *Quarterly Journal of Economics*, 129. 1, 15

Laibson, D. (1997): "Golden eggs and hyperbolic discounting," *The Quarterly Journal of Economics*, 443–477. 10

Piketty, T. (2014): "Capital in the Twenty-First Century," . 1

Stokey, N. L. and E. Lucas Jr, Robert E with Prescott (1989): "Recursive Methods in Economic Dynamics," *Harvard University Press, Cambridge.* 12

Uzawa, H. (1961): "Neutral inventions and the stability of growth equilibrium," *The Review of Economic Studies*, 28, 117–124. 22