Problem Set 1

Answer Key

Part I

Math

Exercise 3.3.c. Show that the following is a metric space: Let S be the set of all continuous, strictly increasing functions on $[a, b]$ with $\rho(x, y) = \max_{a \le t \le b} |x(t) - y(t)|$.

Solution: Take three arbitrary functions $x, y, z \in S$. The first two properties follow from the definition of absolute value. To prove the triangle inequality note that

$$\begin{aligned}
\rho(x,z) &= \max_{a\le t\le b} |x(t) - z(t)| \\
&= \max_{a\le t\le b} |x(t) - y(t) + y(t) - z(t)| \\
&\le \max_{a\le t\le b} (|x(t) - y(t)| + |y(t) - z(t)|) \\
&\le \max_{a\le t\le b} |x(t) - y(t)| + \max_{a\le t\le b} |y(t) - z(t)| \\
&= \rho(x,y) + \rho(y,z)
\end{aligned}$$

Exercise 3.3.d Show that the following is a metric space: Let S be the set of all continuous, strictly increasing functions on $[a, b]$ with $\rho(x, y) = \int_a^b |x(t) - y(t)|dt$.

Solution: Since $x$and y are continuous and strictly increasing functions it is clear that $\rho(x, y) \ge 0$ with equality if and only if $x = y$. $\rho(x, y) = \rho(y, x)$ follows from the absolute value function. To see the triangle inequality holds take three arbitrary functions $x, y, z \in S$.

$$\begin{aligned}
\rho(x,z) &= \int_a^b |x(t) - z(t)|dt \\
&= \int_a^b |x(t) - y(t) + y(t) - z(t)|dt \\
&\le \int_a^b (|x(t) - y(t)| + |y(t) - z(t)|)dt \\
&\le \int_a^b |x(t) - y(t)|dt + \int_a^b |y(t) - z(t)|dt \\
&= \rho(x,y) + \rho(y,z)
\end{aligned}$$

Exercise 3.4.e Show the following is a normed vector space. Let S be the set of all continuous functions on $[a, b]$, with $\| x \| = \sup_{a \le t \le b} |x(t)|$. (This space is called $C([a, b])$.)

Solution: $\| x \| \ge 0$ with equality if and only if $x = \theta$ follows from absolute value and supremum. $\| \alpha x \| = \sup_{a\le t\le b} |\alpha x(t)| = |\alpha| \sup_{a\le t\le b} |\alpha x(t)| = |\alpha| \| x \|$. Finally the triangle inequality

$$\begin{aligned}
\| x + y \| &= \sup_{a\le t\le b} |x(t) + y(t)| \\
&\le \sup_{a\le t\le b} (|x(t)| + |y(t)|) \\
&\le \sup_{a\le t\le b} |x(t)| + \sup_{a\le t\le b} |y(t)| \\
&= \| x \| + \| y \|
\end{aligned}$$

Exercise 3.4.f Show the following is a normed vector space. Let S be the set of all continuous functions on $[a, b]$, with $\| x \|= \int_a^b |x(t)|dt$.

Solution: $\| x \| \geq 0$ with equality if and only if $x = \theta$ follows from absolute value function. $\| \alpha x \|= \int_a^b |\alpha x(t)|dt = \| x \|= |\alpha| \int_a^b |x(t)|dt = |\alpha| \| x \|$. The triangle inequality then follows from

$$\begin{aligned}
\| x + y \| &= \int_a^b |x(t) + y(t)|dt \\
&\leq \int_a^b (|x(t)| + |y(t)|)dt \\
&\leq \int_a^b |x(t)|dt + \int_a^b |y(t)|dt \\
&= \| x \| + \| y \|
\end{aligned}$$

Exercise 3.10. Consider the differential equation and the boundary condition $dx(s)/ds = f[x(s)]$, all $s \geq 0$, with $x(0) = c \in \mathbb{R}$. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and for some $B > 0$ satisfies the Lipschitz condition $|f(a) - f(b)| \leq B|a - b|$, all $a, b \in \mathbb{R}$. For any $t > 0$, consider $C[0, t]$, the space of bounded continuous functions on $[0, t]$, with sup norm. Recall from Theorem 3.1 that this space is complete.

a. Show that the operator T defined by

$$(Tv)(s) = c + \int_0^s f[v(s)]dz, \;\; 0 \leq s \leq t,$$

maps $C[0, t]$ into itself. That is, show that if v is bounded and continuous on $[0, t]$, then so is Tv.

Solution: Since v is bounded, the continuous function f is bounded, say by M, on $[- \| v \|, + \| v \|]$. Hence

$$|(Tv)(s)| \leq |c| + sM$$

so Tv is bounded on $[0, t]$. Since

$$\int_0^s f[v(z)]dz$$

is continuous for all f, Tv is continuous.

b. Show that for some $\tau > 0$, T is a contraction on $C[0, \tau]$

Solution: Let $w, v \in C(0, t)$, and let B be their common bound. Note that

$$\begin{aligned}
|Tv(s) - Tw(s)| &\leq \int_0^s |f(v(z)) - f(w(z))|dz \\
&\leq \int_0^s B|v(z) - w(z)|dz \\
&\leq Bs \| v - w \|
\end{aligned}$$

Choose $\tau = \beta/B$, where $0 < \beta < 1$, then $0 \leq s \leq \tau$ implies that $Bs \| v - w \| \leq \beta \| v - w \|$.

Problem 2.

a. Suppose $p \geq 1$ is a real number and consider the following space and norm

$$\begin{aligned}
S &= \left\{ x = (x_1, x_2, ...) | x_i \in \mathbb{R}, \forall i : \sum_{i=1}^{\infty} |x_i|^p < \infty \right\} \\
\| x \|_p &= \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}, \forall x \in S
\end{aligned}$$

S is called the $\ell^p - space$ and $\| \, . \, \|_p$ is called the p-norm. It can be easily shown that $(\ell^p, \| \, . \, \|_p)$ is a normed vector space. Show that $(\ell^p, \| \, . \, \|_p)$ is complete.

Solution: Suppose $a^1, a^2, a^3...$ is a Cauchy sequence in ℓ^p. Note, each term a^k in the sequence is a point in ℓ^p, and so itself a sequence

$$a^k = (a_1^k, a_2^k, ...)$$

Now, to say that $(a^k)_{k=1}^{\infty}$ is a Cauchy sequence in ℓ^p is precisely to say that

$$\forall \epsilon > 0 \, \exists K \in \mathbb{N} \; s.t. \; \forall k, m \geq K, \; \| \, a^k - a^m \, \|_p^p < \epsilon^p.$$

That is for a given $\epsilon > 0$ and sufficiently large k, m, we have

$$\sum_{n=1}^{\infty} |a_n^k - a_n^m|^p < \epsilon^p$$

The above series has all non-negative terms, therefore it is an upper bound for any fixed $a_{n_0} \in \mathbb{R}$. Hence it must be that $(a_{n_0}^k)_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space, there is a limit $a_{n_0} \in \mathbb{R}$ to this sequence. Consider all such limits for all n_0 and call the resulting sequence a. That is to say, a^k converges point-wisely to a candidate sequence a.

We need to show that the candidate sequence is in fact lives in ℓ^p and the a^kconverges to a in ℓ^p sense, not just point-wisely. Since the sequences in ℓ^p are bounded and a^k converges to a point-wisely, the norm of a has to be finite. Since it is a real sequence, and it is bounded a lives in ℓ^p.

Lets show $a^k \to a$. Let $\epsilon > 0$. Since $a \in \ell^p$ there is an $N_1 \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} |a_n|^p \leq \epsilon^p$$

Moreover, since a^k is a Cauchy sequence we know there is an $N_2 \in \mathbb{N}$, so that $k.m \geq N_2$ implies $\| \, a^k - a^m \, \|_p < \epsilon$. Let $N = \max\{N_1, N_2\}$. Then

$$\sum_{n=1}^{\infty} |a_n|^p \leq \epsilon^p \; and \; \| \, a^N - a^k \, \|_p < \epsilon \; \forall k \geq N$$

Now the sequence a^N is in ℓ^p therefore we can choose N' large enough so that

$$\sum_{n=N'}^{\infty} |a_n^N|^p \leq \epsilon^p$$

Note that we can always increase N' and maintain this estimate, so we are free to choose $N' \geq N$. Since $a_n^k \to a_n$ for each fixed n, we can choose K_1 so that $|a_1^k - a_1| \leq \epsilon^p / N'$ for $k \geq K_1$. Likewise we can choose $K_2, K_3 ... K_{N'}$. Letting $K = \max\{K_1, K_2 ... K_{N'}\}$ we have

$$|a_n^k - a_n| < \epsilon^p / N' \; \forall k \geq K \; and \; n \leq N'$$

Now, for any $k \geq K$, break up $b = a^k - a$ as follows

$$(b_n)_{n=1}^{\infty} = (b_n)_{n=1}^{N'-1} + (b_n)_{n=N'}^{\infty}$$

The triangle inequality for the $p - norm$ then gives

$$\| \, a^k - a \, \|_p \leq \left(\sum_{n=1}^{N'-1} |a_n^k - a_n|^p \right)^{1/p} + \left(\sum_{n=N'}^{\infty} |a_n^k - a_n|^p \right)^{1/p}$$

We have already established for the first term for $k \geq K$

$$\left(\sum_{n=1}^{N'-1} |a_n^k - a_n|^p \right)^{1/p} \leq \left(\sum_{n=1}^{N'-1} \frac{\epsilon^p}{N'} \right)^{1/p} = \left(\frac{N-1}{N} \right)^{1/p} \epsilon < \epsilon$$

For the second term, we use the triangle inequality for the $\ell^p - norm$ restricted range $n > N'$ to get

$$\left(\sum_{n=N'}^{\infty} |a_n^k - a_n|^p\right)^{1/p} \leq \left(\sum_{n=N'}^{\infty} |a_n^k|^p\right)^{1/p} + \left(\sum_{n=N'}^{\infty} |a_n|^p\right)^{1/p}$$

Since $N' \geq N$ the second term here is also $< \epsilon$. So summing up to parts we have

$$\| a^k - a \|_p \leq 2\epsilon + \left(\sum_{n=N'}^{\infty} |a_n^k|^p\right)^{1/p}$$

whenever $k \geq K$. So we need to show this final term is small. Here we make one more decomposition: $a_n^k = a_n^k - a_n^N + a_n^N$, and applying triangle inequality once again

$$\left(\sum_{n=N'}^{\infty} |a_n^k|^p\right)^{1/p} \leq \left(\sum_{n=N'}^{\infty} |a_n^k - a_n^N|^p\right)^{1/p} + \left(\sum_{n=N'}^{\infty} |a_n^N|^p\right)^{1/p}$$

The first of these terms is a sum of non-negative terms over $n \geq N'$, so it is bounded above by the sum over $n \geq 1$ which is equal to $\| a^k - a^N \|_p$, which is $< \epsilon$. The second term is also $< \epsilon$ as was established before. Hence we have shown that

$$\forall \epsilon > 0, \ , \exists K \in \mathbb{N}, \ s.t. \ \forall k \geq K \ we \ have \ \| a^k - a \|_p \leq 4\epsilon$$

Since ϵ was arbitrary we have shown that $a^k \to a$ in ℓ^p. This concludes that ℓ^p is complete.

b. Show that the normed vector space given in exercise 3.4.f is not complete. You have to provide an example. Show why that example does not work for 3.4.e.

Solution: Consider the following piece-wise linear function

$$x_n(t) \quad = \quad \begin{cases} 1 & 0 \leq t \leq \frac{1}{2} \\ 1 - n(t - \frac{1}{2}) & \frac{1}{2} \leq t \leq \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} \leq t \leq 1 \end{cases}$$

This is a Cauchy sequence because for all n, m

$$\| x_n - x_m \| = \int_0^1 (x_n(t) - x_m(t))dt \leq \frac{1}{n} \to 0$$

Assume there exists a limit $x_n \to x$. As $n \to \infty$

$$\int_0^{1/2} |x(t) - x_n(t)|dt \leq \int_0^1 |x(t) - x_n(t)|dt \to 0$$

which means $x(t) = 1$ on $[0, 1/2]$. And similarly

$$\int_{1/2}^1 |x(t) - x_n(t)|dt \leq \int_0^1 |x(t) - x_n(t)|dt \to 0$$

Hence $x(t) = 0$ on $[1/2, 1]$ which is a contradiction as $x(t)$ is not a continuous function. Hence we have a continuous function, which is a Cauchy under the integral norm, converging something out of the set S. Therefore S is not complete with the integral norm. This example does not work as the piece-wise function does not constitute a Cauchy sequence under the sup-norm i.e.

$$\sup |x_n(t) - x_m(t)| = \frac{m - n}{n}$$

which does not always converge to zero for any N we can choose $m = 2n$.

Problem 3. Consider $\alpha_i, \beta_i \in \mathbb{R}_+, i = 1, 2$. Suppose we have

$$\alpha_i^2 + \beta_i^2 < \frac{1}{100}, i = 1, 2$$

Show that the following system of equations has a unique solution in $[0, 1]^2$:

$$\begin{aligned} \alpha_1 x^9 + \beta_1 y^9 &= x \\ \alpha_2 x^9 + \beta_2 y^9 &= y \end{aligned}$$

Hint: Use contraction mapping theorem.

Solution: First let's establish that the given maps are self maps. Since $\alpha_i m \beta_i \in R_+$ and $\alpha_i^2 + \beta_i^2 < 1/100$ it must be that $0 \leq |\alpha_i|, |\beta_i| < 1/10$. Given that $x, y \in [0, 1]$ we have

$$0 \leq \alpha_i x^9 + \beta_i y^9 \leq \frac{1}{10} x^9 + \frac{1}{10} y^9 \leq \frac{1}{5}$$

Hence the maps are self maps at the domain $[0, 1]$. Next, we need to show that maps are contractions. Firstly, observe that

$$x^9 - y^9 = (x - y)(x^2 + xy + y^2)(x^6 + x^3 y^3 + y^6) \leq 9|x - y|$$

Hence for any $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

$$\begin{aligned} \| Tz_1 - Tz_2 \|^2 &= \sum_{i=1,2} |\alpha_i (x_1^9 - x_2^9) + \beta_i (y_1^9 - y_2^9)|^2 \\ &\leq \sum_{i=1,2} 81\alpha_i |x_1 - x_2| + 81\beta_i |y_1 - y_2||^2 \\ &= 81[(\alpha_1^2 + \alpha_2^2)|x_1 - x_2|^2 + (\beta_1^2 + \beta_2^2)|y_1 - y_2|^2] \\ &\leq 81 max(\alpha_1^2 + \alpha_2^2, \beta_1^2 + \beta_2^2)[|x_1 - x_2|^2 + |y_1 - y_2|^2] \\ &\leq \frac{81}{100}[|x_1 - x_2|^2 + |y_1 - y_2|^2] \end{aligned}$$

Therefore T is a contraction mapping and has a unique fixed point in $[0, 1]$.

Problem 4. Consider $f : \mathbb{R} \to \mathbb{R}$ to be a differentiable function with a continuous derivative. Prove or disprove that if $|f'(x)| > 2, \forall x \in \mathbb{R}$, then there exists a unique $x \in \mathbb{R}$ s.t. $f(x) = x$.

Solution: Since $|f'(x)| > 2$ for all x it can be either $f'(x) > 2$ or $f'(x) < -2$ since function cannot be continuously differentiable with $|f'(x)| > 2$ and not monotone. Assume $f'(x) > 2$. Choose some x'. If $x' > f(x')$ then there exists x'' such that $x'' < f(x'')$ because as x decreases $f(x)$ decreases at a faster rate and $f(x') - x$ cannot be at infinities (then derivative condition would not be satisfied). After finding such bounds intermediate value theorem guarantees the fixed point exists. Such bounds can be found if $x' < f(x')$ by increasing x'. Such a fixed point has to be unique. Assume for contradiction there are two fixed points. Then by Mean Value Theorem there exists a point c such that $f'(c) = \frac{f(x_1 - f(x_2)}{x_1 - x_2} = 1$ which is a contradiction. The cases where $f'(x) < -2$ symmetrically follows.

Part II

Problem 5. Sequential Trading vs Time-0 Trading Here you are asked to show that if we allow for sequential trading in the model in class, the outcome will be the same. Suppose that the agents, preferences and technology are the same as those discussed in the class.

Trading is organized as follows: in each period, t, households can purchase any amount of a financial asset called IOU- also known as *Arrow Security*- which entitles them to a unit of the consumption good in period $t + 1$. Thus their budget constraint is given by

$$p_{c,t} c_t^j + p_{x,t} x_t^j + q_t a_t^j \leq w_t (e_t^j - l_t^j) + r_t k_t^j + p_{c,t} a_{t-1}^i$$

where q_t is the price of Arrow securities. To be more precise, there are 5 spot markets now that open each day: markets for consumption goods, investment goods, labor, rental of capital, IOUs. Now, each household maximizes his/her utility subject to the above budget constraint and another constraint of the form

$$a_t \geq -A$$

where A is some large positive number.

(Side Note: The constraint above requires some explanation. The issue is that if the household can freely borrow, he/she might run Ponzi schemes. This is not an issue if horizon is finite since there is always a last period in which the household has to pay up. With infinite horizon, and without any constraint, in theory it is possible to have a Ponzi scheme. A scheme in which the household borrows and then pays the interest and the principle with new borrowing and does this forever. Somehow we need to impose a constraint so that such Ponzi schemes are not possible. The above is one. They typical way of doing it is to impose the following constraint

$$\lim_{t\to\infty} q_t a_t^j \geq 0$$

i.e., the total market value of your assets have to be finite. Note also that at the optimum this constraint won't be binding so we don't have to worry about it!)

a. What should be the feasibility constraint in this new economy? In other words, what is market clearing for the market for Arrow securities? Explain your answer

Solution: Market clearing for the Arrow securities is $\sum_{j=1}^{j} a_t^j = 0$. Note that, the assets are not part of the production. They are just trading of the claims agents have over future consumption goods. Therefore, for every Arrow Security some one buys, there has to be a seller that is giving up his/her claim in the future.

b. Define a competitive equilibrium for this economy.

Solution: The competitive equilibrium for this economy is allocations for households$\{c_t^j, l_t^j, k_t^j, x_t^j, a_t^j\}$ and firms $\{k_t^{i_c}, k_t^{i_k}, n_t^{i_c} n_t^{i_k}\}$ together with prices $\{p_{c,t}, p_{x,t}, q_t, w_t, r_t\}$ for all t such that

- Given prices $\{p_{c,t}, p_{x,t}, q_t, w_t, r_t\}$, the allocation for households $\{c_t^j, l_t^j, k_t^j, x_t^j, a_t^j\}$ solves the following problem

  $$\max U^j(\{c_t^j, l_t^j\}_t$$

  subject to the budget constraint given in the question and law of motion for capital: $k_{t+1}^j = k_t^j(1-\delta) + x_t^j$.

- Given prices $\{p_{c,t}, p_{x,t}, q_t, w_t, r_t\}$ allocations for consumption and capital producing firms maximize their profit

  $$p_{c,t} F^{i_c}(k_t^{i_c}, l_t^{i_c}) - w_t n_t^{i_c} - r_t k_t^{i_c}$$

  and

  $$p_{x,t} F^{i_k}(k_t^{i_k}, l_t^{i_k}) - w_t n_t^{i_k} - r_t k_t^{i_k}$$

- And markets clear

$$\begin{aligned}
\sum_{j=1}^{J} c_t^j &= \sum_{i=1}^{I} F^{i_c}(k_t^{i_c}, n_t^{i_c}) \\
\sum_{j=1}^{J} x_t^j &= \sum_{i=1}^{I} F^{i_k}(k_t^{i_k}, n_t^{i_k}) \\
\sum_{j=1}^{J} a_t^j &= 0 \\
\sum_{j}^{J} e_t^j - l_t^j &= \sum_{j=1,2} \sum_{i=1}^{I} n_t^{i_j} \\
\sum_{j}^{J} k_t^j &= \sum_{j=1,2} \sum_{i=1}^{I} k_t^{i_j}
\end{aligned}$$

c. Show that any allocation resulting from any competitive equilibrium in the Arrow-Debreu economy, i.e., time-0 trading is also part of an allocation in the CE of the economy with sequential markets.

Solution: Firms problem is static so nothing changes there. We need to show that the households problem also similar. Define recursively

$$a_t^j = (w_t(e_t^j - l_t^j) + r_t k_t^j + q_{t-1}a_{t-1}^j - p_{c,t}c_t^j - p_{x,t}x_t^j)/q_t$$

starting from $a_-^j = 0$ and $q_t = p_{c,t}$. Therefore the we have the budget sets in two problems identical. Since the allocation under AD solves this problem then the same allocation also solves SE.

d. Consider the competitive equilibrium in the sequential market economy described above. What is the relationship between prices of investment good, $p_{x,t}$, and rental capital rate of capital, r_t? What is the relationship between rental rate of capital r_t, and the price of Arrow securities , q_t? *Hint:*Think about arbitrage!!!

Solution: Since we know the two economies are equivalent we can just derive the relationship in AD economy.

$$\begin{aligned}
\sum_{t=0}^{\infty} p_{c,t}c_t + p_{x,t}x_t &\leq \sum_{t=0}^{\infty} r_t k_t + w_t l_t \\
k_{t+1} &= x_t + (1-\delta)k_t \\
\sum_{t=0}^{\infty} p_{c,t}c_t + p_{x,t}(k_{t+1} - (1-\delta)k_t) &= \sum_{t=0}^{\infty} r_t k_t + w_t l_t
\end{aligned}$$

Optimality with respect to k implies

$$p_{x,t} = r_{t+1} + (1-\delta)p_{x,t+1}$$

One can also easily come up with an arbitrage strategy. If we increase investment at t by $1/p_{x_t}$ and reduce investment by $(1-\delta)/p_{x,t}$ tomorrow to keep future capital unchanged: this should nor increase or decrease the budget as capital does not enter the utility function. In other words we must have

$$-1 + \frac{r_{t+1}}{p_{x,t}} + \frac{(1-\delta)p_{x,t+1}}{p_{x,t}} = 0$$

which yields the same equation as above.

As for price of Arrow Securities: We can use a dollar to buy $1/q_t$ units of AS today which entitles the consumer to $1/q_t$ units of consumption good tomorrow. Use a dollar to buy $1/p_{x,t}$ units of investment good today - reduce investment by $(1-\delta)\frac{1}{p_{x,t}}$ tomorrow to keep everything else the same in the future. This leads to a total income of in period $t+1$

$$r_{t+1}\frac{1}{p_{x,t}} + \frac{(1-\delta)p_{x,t+1}}{p_{x,t}}$$

which can be used to purchase

$$\frac{1}{p_{c,t+1}}\left[r_{t+1}\frac{1}{p_{x,t}} + \frac{(1-\delta)\,p_{x,t+1}}{p_{x,t}}\right]$$

at $t+1$. By first part the term in brackets is 1. Therefore

$$q_t = p_{c,t+1}$$

Problem 6. Derive equation (2) in notes.

Solution: The profit of the firms in each period is given by

$$p_{c,t}F^{i_c}(k_t^{i_c}, n_t^{i_c}) - w_t n_t^{i_c} - r_t k_t^{i_c}$$

and

$$p_{x,t}F^{i_x}(k_t^{i_x}, n_t^{i_x}) - w_t n_t^{i_x} - r_t k_t^{i_x}$$

Summing profits over time and individuals gives the desired equation as shares$\theta^j_{i_c}$ and $\theta^j_{i_k}$ across households sum to one.

$$\begin{aligned}
\sum_{j=1}^{J} \Pi^j &= \sum_{j=1}^{J} \sum_{i_c=1}^{I_c} \theta^j_{i_c} \sum_{t=0}^{\infty} p_{c,t} F^{i_c}(k_t^{i_c}, n_t^{i_c}) - w_t n_t^{i_c} - r_t k_t^{i_c} \\
&+ \sum_{j=1}^{J} \theta^j_{i_k} \sum_{i_k=1}^{I_k} \sum_{t=0}^{\infty} p_{x,t} F^{i_x}(k_t^{i_x}, n_t^{i_x}) - w_t n_t^{i_x} - r_t k_t^{i_x} \\
&= \sum_{i_c=1}^{I_c} \sum_{t=0}^{\infty} \sum_{j=1}^{J} \theta^j_{i_c} \left[p_{c,t} F^{i_c}(k_t^{i_c}, n_t^{i_c}) - w_t n_t^{i_c} - r_t k_t^{i_c} \right] \\
&+ \sum_{i_k=1}^{I_k} \sum_{t=0}^{\infty} \sum_{j=1}^{J} \theta^j_{i_k} \left[p_{x,t} F^{i_x}(k_t^{i_x}, n_t^{i_x}) - w_t n_t^{i_x} - r_t k_t^{i_x} \right] \\
&= \sum_{i_c=1}^{I_c} \sum_{t=0}^{\infty} \left[p_{c,t} F^{i_c}(k_t^{i_c}, n_t^{i_c}) - w_t n_t^{i_c} - r_t k_t^{i_c} \right] \\
&+ \sum_{i_k=1}^{I_k} \sum_{t=0}^{\infty} \left[p_{x,t} F^{i_x}(k_t^{i_x}, n_t^{i_x}) - w_t n_t^{i_x} - r_t k_t^{i_x} \right]
\end{aligned}$$

which is the desired claim.

Problem 7. **Economics of Netflix** In our model, we assume that each person is endowed with a certain level of leisure whose values fixed. As we discussed in class, services like entertainment, can potentially increase value of leisure. Does our framework capture this? Think of a static economy in which there are no investment goods and there is only consumption good. If not, write an alternative model that does.

Solution: Our framework can capture the increasing value of leisure due to entertainment services to the extend that we can make their consumption complimentary to leisure. To see this, introduce services like entertainment as consumption goods. Then we need value of leisure to depend on the amount of entertainment consumption one has, i.e.

$$U(c, c_e, l)$$

with $\frac{\partial U(c,c_e,l)}{\partial l \partial c_e} > 0$ i.e. marginal value of leisure is an increasing function of the amount of entertainment goods. Hence the value of leisure depends on whether you have Netflix or not. Also not that with this formulation the value of having entertainment services also depends on whether you have leisure or not.

Problem 8. Aggregation in an endowment economy Consider an endowment economy populated by two individuals; one in which there is no production of any goods. Each individual simply has certain number of "apples" in each period. In particular, suppose that each individual, $j = 1, 2$, has the following vector of endowments:

$$\begin{aligned}
\omega^1 &= (2, 0, 2, 0, ...) \\
\omega^2 &= (0, 2, 0, 2, ...)
\end{aligned}$$

That is individual 1 has endowments of 0 in even periods and individual 2 has endowments of 0 in odd periods. We refer to the endowment of individual j in period t by ω_t^j. Suppose further that individual preferences are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t^j)$$

where c_t^j is the consumption of apples in period t by individual j. The period utility function, u, is strictly increasing and strictly concave. Suppose that these individuals trade these goods at time-0, i.e. we want to characterize Arrow-Debreu equilibrium.

a. Define an Arrow-Debreu equilibrium for this economy.

Solution: Arrow-Debreu equilibrium is allocations $\{c_t^j\}$ and prices $\{p_t\}$ such that , given prices $\{p_t\}$

- Allocation $\{c_t^j\}$ solves individual $j's$ maximization problem

$$\max_{c_t^j} \sum_{t=0}^{\infty} \beta^t u(c_t^j)$$

subject to

$$\sum_{t=0}^{\infty} p_t c_t^j \leq \sum_{t=0}^{\infty} p_t \omega_t^j$$

Markets clear

$$\sum_{j=1}^{2} c_t^j = \sum_{j=1}^{2} \omega_t^j \ \forall t$$

b. Show that the CE in part **a.** is Pareto optimal.

Solution: Assume the allocation $\{c_t^j\}$ is not Pareto optimal. Then there exists another feasible allocation $\{c*_t^j\}$ such that it gives one of the households (say individual 1) strictly higher utility without decreasing the utility of other. Note that utility functions are strictly increasing. Therefore, since$\{c_t^j\}$ maximizes the individual 1's problem, $\{c*_t^j\}$ allocation should be out of the budget set:

$$\sum_{t=0}^{\infty} p_t c_t^1 > \sum_{t=0}^{\infty} p_t \omega_t^1$$

Since the other agent cannot get worse-off, we must also have

$$\sum_{t=0}^{\infty} p_t c_t^2 \geq \sum_{t=0}^{\infty} p_t \omega_t^2$$

Summing these two constraint together we obtain

$$\sum_{t=0}^{\infty} p_t (c_t^1 + c_t^2) > \sum_{t=0}^{\infty} p_t (\omega_t^1 + \omega_t^2)$$

which violates the feasibility constraint, a contradiction. Therefore A-D equilibrium in part **a.** is Pareto optimal.

c. As we have discussed in the class, a Pareto optimal allocation in this environment is the solution of a programming problem of the following form

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \alpha \sum_{t=0}^{\infty} \beta^t u(c_t^1) + (1-\alpha) \sum_{t=0}^{\infty} \beta^t u(c_t^2)$$

subject to

$$\sum_{j=1}^{2} c_t^j = \sum_{j=1}^{2} \omega_t^j \ \forall t$$

for some $\alpha \in [0, 1]$. Suppose that $u(c) = log(c)$. Using Kuhn-Tucker methods characterize the set of Pareto optimal allocation.

Solution: Rewriting the problem we have

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [\alpha log(c_t^1) + (1-\alpha) \sum_{t=0}^{\infty} log(c_t^2)]$$

subject to

$$\sum_{j=1}^{2} c_t^j = 2 \ \forall t$$

Let λ be the Lagrangian on the feasibility constraint. Taking the first order conditions we get

$$\begin{aligned}\frac{\alpha}{c_t^1} &= -\lambda \\ \frac{1-\alpha}{c_t^2} &= -\lambda\end{aligned}$$

Solving them together for we get $c_t^1 = 2\alpha$ and $c_t^2 = 2(1-\alpha)$

d. For what values of α can you construct an AD CE as defined in part **a.**?

Solution: Let's first characterize the allocation in AD equilibrium. Each individual solves the following problem

$$\max_{c_t^j} \sum_{t=0}^{\infty} \beta^t log(c_t^j)$$

subject to

$$\sum_{t=0}^{\infty} p_t c_t^j \leq \sum_{t=0}^{\infty} p_t \omega_t^j = I^j$$

Taking the first order conditions for c_t and c_{t+1} we have

$$\frac{\beta^t}{c_t^j} = \lambda p_t$$

and

$$\frac{\beta^{t+1}}{c_{t+1}^j} = \lambda p_{t+1}$$

solving them together leads to following optimality condition

$$\beta p_t c_t^j = p_{t+1} c_{t+1}^j$$

Starting from $p_0 c_0$ and substituting the other terms in terms of the period-zero good we get

$$\begin{aligned}p_0 c_0^j + \beta p_0 c_0^j + \beta^2 p_0 c_0^j ... &= I^j \\ c_0^j &= \frac{I^j(1-\beta)}{p_0}\end{aligned}$$

Iterating over c_0 using the optimality condition we get following demand functions.

$$c_t^j = \frac{I^j(1-\beta)\beta^t}{p_t}$$

Given the demand functions market clearing requires

$$\begin{aligned}c_t^1 + c_t^2 &= 2 \\ \frac{I^1(1-\beta)\beta^t}{p_t} + \frac{I^2(1-\beta)\beta^t}{p_t} &= 2 \\ 2\sum_{t=0}^{\infty} p_t &= \frac{2p_t}{(1-\beta)\beta^t} \\ \frac{p_t}{\sum_{t=0}^{\infty} p_t} &= (1-\beta)\beta^t\end{aligned}$$

Normalizing $p_0 = 1$ we have the solution for prices as $p_t = \beta^t$. Given this prices respective income streams for individuals are

$$\begin{aligned}I^1 &= 2 + 2\beta^2 + 2\beta^4 ... = \frac{2}{1-\beta^2} \\ I^2 &= 2\beta + 2\beta^3 + 2\beta^5 ... = \frac{2\beta}{1-\beta^2}\end{aligned}$$

Plugging in the prices and the respective incomes back into demand functions we find $c_t^1 = \frac{2}{1+\beta}$ and $c_t^2 = \frac{2\beta}{1+\beta}$. Hence we find that the programming problem in part **c.** and AD CE coincide at $\alpha = \frac{1}{1+\beta}$.