Problem Set 2
Answer Key

December 2, 2016

Problem 1. Dynamic Programming Consider the following planning problem in an economy with leisure:

$$\max \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1-n_t\right)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &\leq f\left(k_t, n_t\right) + (1-\delta)\, k_t \\ c_t, k_{t+1}, n_t &\geq 0, \forall t \geq 0 \\ k_0 &: \text{ given} \end{aligned}$$

a. Write this problem in canonical form. You should only use one state variable.

Solution. Assuming $u(.)$ is increasing in c we have the feasibility constraint holding with equality,

$$c_t = f\left(k_t, n_t\right) + (1-\delta)\, k_t - k_{t+1}$$

Substituting this constraint back into the planning problem we have

$$v(k_0) = \max_{k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t [u(f\left(k_t, n_t\right) + (1-\delta)\, k_t - k_{t+1}, 1-n_t)]$$

with the positivity constraints. The only state variable is the capital holdings as it carries the required information the agent needs to know this period. This is not true for the other variable n_t as the agent does not need to know how much he/she worked last period. All the necessary information from past is summarized by the capital holdings.

b. Write the Bellman equation associated with the above canonical form. Provide sufficient conditions on u, β, δ so that the solution of Bellman equation is the same as the solution of the sequence problem. State the theorems that you use.

Solution. The associated Bellman equation with the given sequence problem is

$$V(k) = \max_{k', n} u(f\left(k, n\right) + (1-\delta)\, k - k', 1-n) + \beta V(k')$$

For this problem to have the same solution to the sequence problem we refer to Theorem 4.3. in SLP which states given the following assumptions solution to the functional equation coincides with the solution to the sequence problem:

Assumption 4.1. $f(k_t, n_t)$ is non-empty, for all $k \geq 0$ and $n \in [0, 1]$.

Assumption 4.2. For all $k_0 \geq 0$ and $\hat{k} \in \Pi(k_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t [u(f(k_t, n_t) + (1 - \delta) k_t - k_{t+1}, 1 - n_t)]$ exists (although it may be plus or minuse infinity) where $\Pi(k_0)$ is the all attainable capital levels considering all possible choices of $n \in [0, 1]$

Tail Condition: $\lim_{n\to\infty} \beta^n v(k_n) = 0$**,** all $(k_0, k_1 ...) \in \Pi(k_0)$, all $k_0 \geq 0$ **.**

c. Assume that $u(c, 1 - n) = \gamma \log(c) + (1 - \gamma) \log(1 - n)$,$\delta = 1$, $f(k, n) = k^\alpha n^{1-\alpha}$. We want to solve the Bellman equation by guess and verify method. Guess that $v(k) = A + B \log(k)$ and find A, B. Then verify that the Bellman equation holds. Find the policy function for future capital. And provide the auxiliary policy functions for consumption and labor supply.

Solution. Plugging in the specified functions back into the Bellman equation from part (b) we have

$$v(k) = \max_{k',n} \gamma \log(k^\alpha n^{1-\alpha} - k') + (1-\gamma) \log(1-n) + \beta v(k')$$

Substituting $v(k')$ with the guess $v(k') = A + B \log(k')$

$$v(k) = \max_{k',n} \gamma \log(k^\alpha n^{1-\alpha} - k') + (1-\gamma) \log(1-n) + \beta A + \beta B \log(k')$$

Taking the first order condition with respect to k' we have the following optimality condition

$$\frac{\gamma}{k^\alpha n^{1-\alpha} - k'} = \frac{\beta B}{k'}$$

Solving for k' we have

$$k' = \frac{\beta B k^\alpha n^{1-\alpha}}{\gamma + \beta B}$$

Solving for the optimal labor allocation we have

$$\frac{\gamma(1-\alpha) k^\alpha n^{-\alpha}}{k^\alpha n^{1-\alpha} - k'} = \frac{(1-\gamma)}{1-n}$$

Plugging in the optimal capital k' and solving for n yields

$$n^* = \frac{(1-\alpha)(\gamma + \beta B)}{(1-\alpha)(\gamma + \beta B) + (1-\gamma)}$$

Hence we are looking for A and B such that the following equation is satisfied

$$A + Blog(k) = \gamma \log(k^\alpha n *^{1-\alpha} - \frac{\beta B k^\alpha n*^{1-\alpha}}{\gamma + \beta B}) + (1-\gamma) \log(1-n^*) + \beta A + \beta B \log(\frac{\beta B k^\alpha n*^{1-\alpha}}{\gamma + \beta B})$$

Regrouping terms we have

$$A + Blog(k) = \gamma log(\gamma n^{1-\alpha}) + \gamma \alpha log(k) - \gamma log(\gamma + \beta B) + (1-\gamma) \log(1-n^*) + \beta A + \beta B log(\beta B n*^{1-\alpha}) + \beta B \alpha log(k)$$

Hence we have

$$B = \alpha\gamma + \alpha\beta B$$

we solves for

$$B = \frac{\alpha\gamma}{1-\alpha\beta}$$

Plugging in B back into the equation for $n*$ we have

$$n* = \frac{\gamma(1-\alpha)}{\gamma(1-\alpha)+(1-\gamma)(1-\alpha\beta)}$$

Finally substituting $n*$ to the remainder of the function and solving for A yields

$$A = \frac{\gamma}{(1-\beta)(1-\alpha\beta)}[(1-\alpha)log(\frac{\gamma(1-\alpha)}{\gamma(1-\alpha)+(1-\gamma)(1-\alpha\beta)})+(1-\alpha\beta)log(1-\alpha\beta)+\alpha\beta log(\alpha\beta\gamma)]+\frac{1-\gamma}{1-\beta}log$$

Hence we end up an equation of the form $v(k) = A + Blog(k)$ with A and B defined as above; i.e. we verified that the Bellman equation holds. For the policy functions we have

$$n* = \frac{\gamma(1-\alpha)}{\gamma(1-\alpha)+(1-\gamma)(1-\alpha\beta)}$$

$$k*' = \alpha\beta k^{\alpha} n^{1-\alpha}$$

$$c* = (1-\alpha\beta)k^{\alpha} n^{1-\alpha}$$

d. Assume that $\gamma = 1$. So $n(k) = 1$. Starting from $v_0(k) = 0$, apply the operator T defined by the Bellman equation. Calculate the function $T^n v_0$ and show that $T^n v_0 \to v$ where v is the solution of the Bellman equation.

Solution. Plugging in the assumptions for γ and $n(k)$ we have

$$v(k) = \max_{k'} \log(k^{\alpha} - k') + \beta v(k')$$

Define the operator T as

$$Tv_n(k) = \max_{k'} \log(k^{\alpha} - k') + \beta v_n(k')$$

Starting from $v_0(k) = 0$ we have

$$Tv_0(k) = \max_{k'} \log(k^{\alpha} - k') = \alpha log(k)$$

Letting $v_1(k) = \alpha log(k)$ we have

$$T^2 v_0(k) = \max_{k'} \log(k^{\alpha} - k') + \alpha\beta \log(k')$$

Solving for the optimal k' we have

$$k' = \frac{\alpha\beta k^{\alpha}}{1+\alpha\beta}$$

plugging this result back into $Tv_1(k)$ we have

$$\begin{aligned} Tv_0^2(k) &= log(k^\alpha - \frac{\alpha\beta k^\alpha}{1+\alpha\beta}) + \alpha\beta log(\frac{\alpha\beta k^\alpha}{1+\alpha\beta}) \\ &=(1+\alpha\beta)\alpha log(k) + \alpha\beta log(\alpha\beta) - (1+\alpha\beta)log(1+\alpha\beta) \end{aligned}$$

Thus, if we have $v_n\left(k\right) = A_n \log k + B_n$, then

$$Tv_n\left(k\right) = \max_{k'} \log\left(k^\alpha - k'\right) + \beta A_n \log k' + \beta B_n$$

Therefore

$$\frac{1}{k^\alpha - k'} = \frac{\beta A_n}{k'} \rightarrow k' = \frac{\beta A_n}{1+\beta A_n} k^\alpha$$

Replacing in the objective, we have

$$\begin{aligned} v_{n+1}\left(k\right) = Tv_n\left(k\right) =& \alpha \log k - \log\left(1+\beta A_n\right) + \alpha\beta A_n \log k \\ &+ \beta A_n \left[\log\left(\beta A_n\right) - \log\left(1+\beta A_n\right)\right] + \beta B_n \end{aligned}$$

Therefore

$$\begin{aligned} v_{n+1}\left(k\right) &= A_{n+1} \log k + B_{n+1} \\ A_{n+1} &= \alpha\left(1+\beta A_n\right) \\ B_{n+1} &= \beta A_n \log\left(\beta A_n\right) - \left(1+\beta A_n\right)\log\left(1+\beta A_n\right) + \beta B_n \end{aligned}$$

with

$$A_0 = 1, B_0 = 0$$

It can be easily seen that

$$\begin{aligned} A_n &= \alpha + \left(\alpha\beta\right) A_{n-1} = \alpha + \left(\alpha\beta\right)\left[\alpha + \left(\alpha\beta\right) A_{n-2}\right] \\ &= \alpha\left[1 + \left(\alpha\beta\right) + \left(\alpha\beta\right)^2 + \cdots + \left(\alpha\beta\right)^{n-1}\right] \\ \lim_{n\to\infty} A_n &= \frac{\alpha}{1-\alpha\beta} = A \end{aligned}$$

If B_n converges to a limit this limit is given by

$$B = \frac{\beta A \log \beta A - \left(1+\beta A\right)\log\left(1+\beta A\right)}{1-\beta}$$

which is the same as in the closed form above.

Problem 2.

Solve exercise 5.1 in SLP.

Solution. **a.-c.** We need to establish Assumptions 4.1-4.8.

A4.1: Let $\Gamma(x) = [0, f(x)]$. Since by T2 $f(0) = 0$, $0 \in \Gamma(x)$ for all x, hence $\Gamma(x)$ is non-empty for all x.

A4.2: Let $F(x_t, x_{t+1}) = U(f(x_t) - x_{t+1})$. By U3 and the fact that $U : R_+ \to R$, U, and hence F, is bounded below and the result follows from U1.

A4.3: $X = [0, \overline{x}]$ is a convex subset of $\mathbb{R}$. $\Gamma(x)$ is non-empty from part the first part. Given x, $[0, f(x)]$ is compact and $\Gamma(x)$ is compact valued. Since $f(x)$ is continuous, the correspondence $[0, f(x)]$ is continuous.

A.4.4: We showed that F is bounded below. By T1-T3 $f(x_t) - x_{t+1}$ is bounded, and hence by assumption $U2 - U3$ F is bounded above. Hence F is bounded. It is continuous by U2 and T1. $0 < \beta < 1$ by U1.

A4.5: By U3 and T3, $F(., y)$ is a strictly increasing function.

A4.6:Let $x < x'$, then by T3, $f(x) \leq f(x')$, which implies that $[0, f(x)] \subseteq [0, f(x')]$.

A4.7: By T4 $f(x) - y$ is a concave function in (x, y). By U4 this implies that $F(x, y)$ is strictly concave in (x, y).

A4.8: Let $x, x' \in X$, $y \in \Gamma(x)$ and $y' \in \Gamma(x')$. Then $y \leq f(x)$ and $y' \leq f(x')$, which implies, by T4, that

$$\begin{aligned} \theta y + (1-\theta)y' \leq & \quad \theta f(x) + (1-\theta) f(x') \\ \leq & \quad f(\theta x + (1-\theta)x') \end{aligned}$$

d. $v(x)$ is differentiable at x : By Theorems 4.7 and 4.8 and parts b. and c. v is an increasing and strictly concave function. By U5, T5 and $g(x) \in (0, f(x))$, Assumption 4.9 is satisfied. Hence by Theorem 4.11 v is continuously differentiable and

$$v'(x) = F_x[f(x) - g(x)] = U'[f(x) - g(x)]f'(x)$$

A sufficient condition for the interior solution $g(x) \in (0, f(x))$ is

$$\lim_{c \to 0} U'(x) = \infty$$

To see that $g(x) = f(x)$ is never optimal we have

$$\lim_{g(x) \to f(x)} U'(f(x) - g(x)) = \infty$$

while

$$\lim_{g(x) \to f(x)} \beta v'(g(x)) < \infty$$

by the strict concavity of v. Hence the utility from decreasing x today increases much faster than it decreases the value from saving it. Therefore it cannot be $g(x) = f(x)$.

To see that $g(x) = 0$ is not optimal, assume $g(\hat{x}) = 0$ for some $\hat{x} > 0$. Hence it must be that $g(x) = 0$ for all $x < \hat{x}$. But then, for $x < \hat{x}$

$$v(x) \equiv U(f(x)) + \beta \frac{U(0)}{1-\beta}$$

Therefore v is differentiable and

$$v'(x) = U'[f(x)]f'(x)$$

Hence, when $x \to 0$, $v'(x) \to \infty$, and then $g(\hat{x}) = 0$ for $\hat{x}$ is not possible.

To see what happens when this condition fails, notice that at the steady state, we have $g(x^\star) < f(x^\star)$, where $x^\star$ stands for the steady state level of capital. By continuity of g, there is an interval $(x^\star - \epsilon, x^\star)$ such that for any x belonging to that interval, $g(x) < f(x)$. Theorem 4.11 implies that v is differentiable in this range. For any other x, eventually this interval will be reached, or another point interval that implies $g(x) = 0$ or $g(x) < f(x)$. We established above that v is differentiable at those cases, so it must be that v is differentiable everywhere.

e. Let $\beta' > \beta$. Define T' as the operator T using β' as a discount factor instead of β, and v_k as the k^{th} application of this operator.

Applying T' to $v(x, \beta)$ once we obtain $v_1(x, \beta')$ and $g_1(x, \beta')$, where using first order conditions (assuming an interior solution for simplicity), $g_1(x, \beta')$ is defined as the solution y to

$$U'(f(x) - y) = \beta' v(y, \beta').$$

It is clear that the savings function must increase since the right hand side increases from β to β', that is $g_1(x, \beta') > g_1(x, \beta)$, which by Theorem 4.11 in turn implies that

$$v_1'(x, \beta') > v'(x, \beta).$$

By a similar argument, if

$$v_k'(x, \beta') > v_{k-1}'(x, \beta'),$$

then

$$v_{k+1}'(x, \beta') > v_k'(x, \beta'),$$

and

$$g_{k+1}(x, \beta') > g_k(x, \beta').$$

Hence $g_k(x, \beta')$ increases with k. The result then follows from applying Theorem 4.9 to the sequence $\{g_k(x, \beta'\}_{k=0}^{\infty}$ since $g_k(x, \beta') \to g(x, \beta')$.

Problem 3. Show the weak version of Uzawa's theorem as stated in the notes. That is, with capital augmenting technical change, a balanced growth path exists only if production function is Cobb-Douglas.

Hint: You only need to use the feasibility constraint and the definition of BGP.

Solution. The proof uses a continuous time formulation. On a balanced growth path we have the output and capital growing at the same rate [assuming no population growth for simplicity] and $\frac{dlnA}{dt} = g$ by assumption.

$$Y_t = F(A_t K_t, L_t)$$

Assuming the production function is homogeneous of degree 1 we have

$$y_t = F(A_t k_t, 1) = f(A_t k_t)$$

Taking the log of both sides

$$lny_t = lnf(A_t k_t)$$

Differentiating both sides with respect to time t

$$\frac{dlny}{dt} = \frac{\partial lnf(Ak)}{\partial lnk}\frac{dlnk}{dt} + \frac{\partial lnf(Ak)}{\partial lnA}\frac{dlnA}{dt}$$

Since technology is capital augmenting, the elasticity of output with respect to capital and the technology parameter is the same, say η. Hence we have

$$\frac{dlny}{dt} = \eta\frac{dlnk}{dt} + \eta\frac{dlnA}{dt}$$

Since we are on the BGP then $\frac{dlny}{dt} = \frac{dlnk}{dt} = \hat{g}$ we have

$$\hat{g} = \eta\hat{g} + \eta g$$

$$\hat{g} = \frac{\eta g}{1-\eta}$$

Hence for $\hat{g}$ to be constant over time (so that we are on BGP) η has to be constant as well. Which implies the production function to have Cobb-Douglas form.

Problem 4. Balanced Growth Path and Labor Supply In class we defined a balanced growth path as sort of a steady state where all per-capita variables grow at a constant rate. We mostly worked with examples where labor supply was inelastic. Here we try to see under what conditions balanced growth paths exist with endogenous labor supply.

To see this, consider the standard one-sector growth model with elastic labor supply where utility is given by

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t, \ell_t\right)$$

Suppose that population grows at rate n, i.e., $N_{t+1} = (1+n)\, N_t$ and that there is Harrod-neutral productivity growth, i.e.,

$$Y_t = F\left(K_t, A_t N_t \left(1-\ell_t\right)\right)$$
$$A_{t+1} = A_t \left(1+g_A\right)$$

Note that we have assumed here that each individual's total time endowment is normalized to 1. As discussed in the notes, suppose that aggregate welfare is given by

$$\sum_{t=0}^{\infty} \beta^t N_t u\left(c_t, 1-\ell_t\right)$$

a. that $u(c,\ell) = \log c + v(\ell)$ where $v(\cdot)$ is a strictly concave and decreasing function with $v'(1) = 0$. Show that this economy has a modified balanced growth path in which all per capita variables grow at a constant rate except leisure which is constant.

Solution. We can define the hat variables

$$\hat{x}_t = \frac{X_t}{A_t N_t}$$

Then

$$\hat{c}_t + (1+n)(1+g)\hat{k}_{t+1} = F\left(\hat{k}_t, 1-\ell_t\right) + (1-\delta)\hat{k}_t$$

Note that intra-temporal Euler equation is given by

$$v'(\ell_t) = A_t N_t F_n (K_t, A_t N_t (1-\ell_t)) \frac{1}{C_t} \Rightarrow v'(\ell_t) = \frac{1}{\hat{c}_t} F_n (K_t, A_t N_t (1-\ell_t))$$

Since F_n depends only on the capital labor ratio, we must have

$$v'(\ell_t) = \frac{1}{\hat{c}_t} F_n \left(\hat{k}_t, 1-\ell_t\right)$$

The LHS of this equation is decreasing while its RHS is increasing in ℓ_t . Thus there is a unique ℓ_t that solves the above. We can thus put this in a Bellman equation and use recursive methods to show that a modified BGP exists where $\hat{k}_t$ and $\hat{c}_t$ are both constant and as a result so is ℓ_t .

b. Show the same thing as above if $u(c,\ell) = \frac{\left(c^\alpha \ell^{1-\alpha}\right)^{1-\sigma}}{1-\sigma}$ where $\alpha \in (0,1)$ and $\sigma \geq 0$.

Solution. The intra-temporal Euler equation is given by

$$(1-\alpha)\frac{\left(c_t^\alpha \ell_t^{1-\alpha}\right)^{1-\sigma}}{\ell_t} = A_t F_n (K_t, A_t N_t (1-\ell_t))\, \alpha \frac{\left(c_t^\alpha \ell_t^{1-\alpha}\right)^{1-\sigma}}{c_t} \Rightarrow \frac{1}{\ell_t} = \frac{\alpha}{1-\alpha}\frac{1}{\hat{c}_t} F_n (K_t, A_t N_t (1-\ell_t))$$

Now a logic similar to part a proves the claim.

c. Some papers in the literature have found it convenient to work with a utility function that exhibits no income effect, i.e.,

$$u(c,\ell) = \log(c + v(\ell))$$

In a simple static model, optimality of leisure or intratemporal Euler equation becomes

$$\frac{u_\ell}{u_c} = w$$

and with these preferences, the above relationship becomes $v'(\ell) = w$. Therefore, labor supply is independent of consumption or how wealthy is the individual - no income or wealth effect. These preferences are referred to as GHH preferences after Greenwood, Hercowitz and Huffman (1988) published in the AER. Show that this economy does not have a modified BGP as defined above.

Solution. Suppose to the contrary that it does. Consider the intra-temporal Euler equation

$$\frac{v'(\ell_t)}{c_t + v(\ell_t)} = A_t F_n\left(K_t, A_t N_t (1-\ell_t)\right) \frac{1}{c_t + v(\ell_t)}$$
$$v'(\ell_t) = A_t F_n\left(\hat{k}_t, 1-\ell_t\right)$$

Now in a modified BGP, the RHS is growing at rate g_A while the LHS is constant. This is a contradiction. Intuitively, without income effect, the marginal benefit of one hour of working is simply the wage while its marginal cost is the marginal utility of leisure. Since in a BGP wages are growing, ℓ_t cannot stay constant.

d. bonus question. For any utility function, the Frisch elasticity of labor supply is defined as the response of labor supply to wages while holding marginal utility from consumption fixed. In other words, it is given by

$$\xi^f = \left.\frac{\partial \log(1-\ell)}{\partial \log w}\right|_{u_c:\text{fixed}}$$

When $u(c,\ell) = \log c - \psi\frac{(1-\ell)^\gamma}{\gamma}$ for some $\psi > 0$ and $\gamma > 1$, it can be easily shown that Frisch elasticity of labor supply is constant and equal to $\frac{1}{\gamma-1}$. From part a above, we know that these preferences gives us a modified BGP. Can you come up with another class of preferences which has a constant Frisch elasticity of labor supply and is consistent with a modified BGP?

Solution. Suppose that

$$u(c,\ell) = \frac{c^{1-\sigma}\left(1+\psi\frac{\sigma-1}{\gamma}(1-\ell)^\gamma\right)^\sigma}{1-\sigma}$$

Then you can check that the Frisch elasticity is constant and the utility function is consistent with BGP.

Problem 5. Solve exercise 15.3, 15.4, 15.5 in Ljunqgvist and Sargent.

Problem 6. Growth and Inequality Consider an economy that is consisted of two individuals that are only different in terms of their time-0 endowment of capital. The rich and the poor! They supply one unit of labor inelastically and their utility functions are given by

$$\sum_{t=0}^{\infty} \beta^t \log\left(c_t^i - \bar{c}\right)$$

Suppose further that consumption and investment goods are produced using the same production function which is a Cobb-Douglas production function $F(k,n) = Ak^\alpha n^{1-\alpha}$. As we have shown in class any competitive equilibrium of this economy is Pareto optimal and we have provided the formulation of the programming problem associated with that.

a. For a given set of welfare weights (α_R, α_P) with $\alpha_R + \alpha_P = 1$, formulate the Pareto optimal problem as recursive problem with aggregate capital as the only state variable.

Solution. The Pareto problem with the associated weights (α_R, α_P) is given by

$$\sum_{t=0}^{\infty} \beta^t [\alpha_R log(c_t^R - \bar{c}) + \alpha_P log(c_t^P - \bar{c})]$$

subject to

$$c_t^R + c_t^P + K_{t+1} \leq AK_t^{\alpha} + (1-\delta)K_t$$

b. Calculate the steady state value of capital stock for this economy for a given set of welfare weights.

Solution. The first order conditions of the Pareto problem are

$$c_t^R : \quad \frac{\beta^t \alpha_R}{c_t^R - \bar{c}} = \lambda_t$$

$$c_t^P : \quad \frac{\beta^t \alpha_P}{c_t^P - \bar{c}} = \lambda_t$$

$$K_{t+1} : \quad \lambda_{t+1}[\alpha A K_{t+1}^{\alpha-1} + (1-\delta)] = \lambda_t$$

using either consumption streams gives the Euler Equation

$$\frac{\beta^t \alpha_i}{c_t^i - \bar{c}} = [\alpha A K_{t+1}^{\alpha-1} + (1-\delta)] \frac{\beta^{t+1}\alpha_i}{c_{t+1}^i - \bar{c}}$$

Evaluated at steady state we have

$$\beta[\alpha A K^{*\alpha-1} + (1-\delta)] = 1$$

$$K^* = \left[\frac{1-\beta(1-\delta)}{\beta\alpha A}\right]^{1/1-\alpha}$$

Note that the steady state level of the aggregate capital does not depend on the Pareto weights assigned by the social planner.

c. How does inequality, as measured by the ratio of consumption between the rich and the poor evolve over time as the economy converges to its steady state?

Solution. Using the optimality conditions regarding consumption distribution we have

$$\frac{\beta^t \alpha_R}{c_t^R - \bar{c}} = \lambda_t = \frac{\beta^t \alpha_P}{c_t^P - \bar{c}}$$

Solving for c_t^P yields the distribution rule

$$c_t^P = \frac{\alpha_p(c_t^R - \bar{c}) + \alpha_R \bar{c}}{\alpha_R}$$

Consumption inequality is given by

$$\frac{c_t^P}{c_t^R} = \frac{\alpha_P c_t^R + (\alpha_R - \alpha_P)\bar{c}}{\alpha_R c_t^R} = \frac{\alpha_P}{\alpha_R} + \frac{(\alpha_R - \alpha_P)\,\bar{c}}{\alpha_R c_t^R}$$

The path for inequality as the economy converges to steady state, depends on the evolution of c_t^R and the difference in the welfare weights. If $\alpha_R > \alpha_P$ and the economy starts with low level of aggregate capital, then inequality grows over time. This is because consumption must grow over time and $c_t^R - \bar{c} = \alpha_R (C_t - \bar{c})$ where C_t is aggregate consumption. That is, even though the rich agent starts a higher consumption, this inequality actually grows over time. The intuition for this can be thought of as something which resembles a poverty trap. The poor need to consume a lot since they are close to their subsistence level $\bar{c}$. This means that they will not save much. The rich on the other hand are far away from their subsistence level and save more. So inequality grows.

d. bonus question: Can you describe the evolution of wealth inequality in this economy. To do this you need to find the welfare weights associated with a competitive equilibrium and then construct wealth in the competitive equilibrium.

Solution. The only difference between agents is their initial capital stocks. The only way this effects the problem is through renting the capital to the firms. The marginal return to capital $F_k = \alpha A k^{1-\alpha} n^{1-\alpha} \geq 0$ for all k with strict inequality whenever $k \neq 0$. Hence the "rich" agent has more wealth. Given preferences are monotone increasing in c this means that the rich agent will consume more. Hence the supporting Pareto weights must be in a way that $\frac{\alpha_R}{\alpha_P} > 1$. Which suggests that rich will consume higher and higher than the "poor" agent during the transition. The consumer budget constraint is given by

$$c_t^i + k_{t+1}^i = (1 + r_t - \delta)\, k_t^i$$

where $r_t = F_K\left(k_t^R + k_t^P, 1\right)$. As a result, we can write

$$k_t^i = \sum_{s=t}^{\infty} \frac{1}{(1 + r_t - \delta) \cdots (1 + r_s - \delta)} c_t^i$$

This suggests that if the economy starts with a low level of capital, then inequality in wealth should be growing as well, since inequality in consumption increases over time.

Exercise 11.3.

Solution

a. The first order necessary conditions for optimality of the planner problem are described by a second order difference equation in (c, k):

$$u_c(c_t) = \beta\left[u_c(c_{t+1})zf'(k_{t+1} + k_t) + 1\right] + \beta^2 u_c(c_{t+2})zf'(k_{t+2} + k_{t+1}),\ \forall t\ k_0 given$$

For sufficiency we impose some appropriate transversality conditions:

$$\lim_{T\to\infty} \beta^T u_c(c_T^*)k_{T+1}^* = 0,$$

$$\lim_{T\to\infty} \beta^{T+1} u_c(c_{T+1}^*)k_T^* f_k(k_T^* + k_{T-1}^*) = 0.$$

b. In steady state, the marginal utility of consumption and the capital stock are constant. The standard growth model has constant returns to scale technology, so that the production function is homogenous of degree 1. The steady state capital stock is uniquely determined by

$$1 = 2\beta z f'(k) + 2\beta^2 z f'(k),$$

or

$$f'(k) = \frac{1}{2z}\frac{1}{\beta + \beta^2}.$$

Existence and uniqueness follow from the monotonicity of f', the Inada condition $\lim_{k\to\infty} f'(k) = 0$, and the additional assumption $\lim_{k\to 0} f'(k) > \frac{1}{2z}\frac{1}{\beta+\beta^2}$.

c. We will show that a cyclical steady state cannot exist be assuming it does and deriving a contradiction. In odd and even periods resp. The SS equations become:

$$1 = \left[\frac{u_c(c^e)}{u_c(c^o)} + \beta\right]\beta z f'(k^0 + k^e),$$

$$1 = \left[\frac{u_c(c^o)}{u_c(c^e)} + \beta\right]\beta z f'(k^0 + k^e).$$

Both equations can only hold simultaneously if and only if $u_c(c^e) = u_c(c^o)$ or $c^e = c^o$. This is a contradiction to the cyclical steady state.

Exercise 11.4.

Solution

a. The three constraints are:

$$c_t + k_{t+1} \leq zf(\kappa_t) - \delta\kappa_t + k_t,$$

$$\kappa_t \leq k_t,$$

$$c_t \leq zf(\kappa_t).$$

Let λ_t, η_t and ν_t be the Lagrange multipliers on the time t resource constraint, capacity constraint non-negativity constraint on investment respectively.

The first order necessary conditions for optimality of the planner problem are described by:

$$u_c(c_t) = \lambda_t + \nu_t, \tag{126}$$

$$\lambda_t = \beta(\lambda_{t+1} + \eta_{t+1}), \tag{127}$$

$$\lambda_t(zf'(\kappa_t - \delta) + \nu_t zf'(\kappa_t) = \eta_t, \tag{128}$$

plus the complementary slackness conditions. For sufficiency, we impose the transversality condition:

$$\lim_{T\to\infty} \beta^T \lambda_T k_{T+1} = 0.$$

When the capacity constraint binds, the economy operates at full capacity, $\eta_t > 0$, and $k_t = \kappa_t$. The budget constraint and capacity utilization equations reduce to the standard ones. The intertemporal marginal rate of substitution is smaller than 1. For log utility, consumption at time t is higher than consumption at time $t-1$ multiplied by the discount factor.

When the capacity constraint does not bind, $\eta_t = 0$ $k_t > \kappa_t$. The intertemporal marginal rate of substitution equals 1. If investment is positive, $\nu_t = 0$ and $zf'(\kappa_t) = \delta$. The last equality says that the economy utilizes machinery up to the point where the marginal productivity equals the depreciation rate. If investment is zero, $\nu_t > 0$, and the marginal product of of utilized capital is lower than the depreciation rate.

b. and c. In steady state, the first order conditions become

$$u_c(c^*) = \lambda + \nu,$$
$$\lambda = \beta(\lambda + \eta),$$
$$\lambda(zf'(\kappa^* - \delta) + \nu zf'(\kappa^*) = \eta.$$

We show that in steady state the economy operates at full capacity: $k^* = \kappa^*$.

Proof: Because of the Inada conditions on u_c, the steady state consumption level is strictly positive. From the budget constraint $c^* \leq zf(\kappa^*) - \delta\kappa^*$. Because $c^* > 0$, $\kappa^* > 0$ and hence $c^* \leq zf(\kappa^*)$. Therefore $\nu = 0$. Second, $\eta > 0$. If η were zero, then $\lambda = \beta(\lambda + \eta)$ only holds for $\lambda = 0$. But that implies that $u_c(c) = 0$ which contradicts a finite consumption level. Therefore $k^* = \kappa^*$.
Combining the first order conditions, we then get $\beta(1 - \delta + zf'(k^*)) = 1$ and $c^* = zf(k^*) - \delta k^*$. The equations pins down a unique steady state capital stock and consumption level. This follows from the monotonicity of f', the Inada condition $\lim_{k\to\infty} f'(k) = 0$ and the additional assumption $\lim_{k\to 0} f'(k) > \frac{1-\beta+\delta}{z}$.

d. In the steady state all countries are operating at full capacity. Hence, in steady state, differences in output per capita are eliminated. In the long-run, the model is inconsistent with the statement. The reason is that there is a unique level of used machines κ^* that satisfies $f'(\kappa^*) = \frac{1-\beta+\delta}{z}$.

Along the path to the steady state, differences in output per capita can arise, and stem from differences in capacity utilization. There is a (weakly) negative relationship between capacity utilization and output per capita. Countries with low capacity utilization have high output per capita and vice versa.

Absent fluctuations in z, countries with a high initial capital stock ($k_0 > \hat{\kappa}$) will not use their capital stock fully, but choose a *constant* level of machines $\hat{\kappa}$ such that $f'(\hat{\kappa}) = \frac{\delta}{z}$. They use capital up to the point where its marginal product equals the depreciation. They do not invest. Note that $\hat{\kappa} > \kappa^* > k_0$. Because utilized capital depreciates, capacity utilization rises over time. Output per capita is constant and at a level $zf(\hat{\kappa})$. At some point τ, full capacity utilization is reached $k_\tau = \kappa_\tau = \hat{\kappa}$. From that point onwards, the capacity constraint starts to bind ($\eta_\tau > 0$), full capacity utilization is maintained ($k_t = \kappa_t,\ t > \tau$) and the number of machines employed, κ_t, gradually decreases to the steady state level κ^*.

Countries with a low initial capital stock ($k_0 < \kappa^*$) operate at full capacity along the transition path $k_t = \kappa_t$. They accumulate capital until $k_t = \kappa^*$.

Exercise 11.5.

Solution

a. Given a uniform initial wealth distribution $\{k_0^i\}_{i=0}^1$, a competitive equilibrium is a feasible allocation $\{c^i, k^i, n^i\}$ for each agent i and a price vector $\{w, r\}$ such that

- Given prices, households maximize the present discounted value of utility streams subject to their budget constraint $c_t^i + k_{t+1}^i \leq (1-\delta+r_t)k_t^i + w_t n_t$, given k_0^i.
- Firms maximize profits
- The labor market and goods market clear.

b. *i.* The first order necessary conditions for the household problem and firm problem imply:

$$u_c^i(c_t^i) = \beta u_c^i(c_{t+1}^i)(zf_k(k_{t+1}, n_{t+1}) + 1 - \delta).$$

Households supply their unit of labor inelastically. Capital and labor are paid their marginal products.
In steady state, consumption is constant for every agent, and so is marginal utility. Therefore, there is a unique steady state capital stock, determined by $zf_k(k,1) = \beta^{-1} - 1 + \delta$. The existence and uniqueness follow from the monotonicity of $f_k(\cdot, 1)$, the Inada condition on f_k and the additional assumption $\lim_{k\to 0} f'(k) > \beta^{-1} - 1 + \delta$. The steady state interest rate $r(k^*)$ can be calculated without info on u_i. Optimal steady state consumption is: $c^* = f(k^*, 1) - \delta k^*$. Economist A is correct.

ii. Economist A is correct again. At any period t, optimality requires equalization of the IMRS of any pair of agents (i, j). Rewriting this equality, we get:

$$\frac{u_c^j\left(c_t^j\left(k_t^j\right)\right)}{u_c^i\left(c_t^i\left(k_t^i\right)\right)} = \frac{u_c^j\left(c_{t+1}^j\left(k_{t+1}^j\right)\right)}{u_c^i\left(c_{t+1}^i\left(k_{t+1}^i\right)\right)}$$

This condition implies that the marginal rate of substitution between consumers (i, j) is constant through time. The initial distribution of capital determines the initial marginal rate of substitution between consumers (i, j). By the equation above, the initial capital stock determines the consumption inequality and the latter stays constant over time.

c. In steady state, the capital stock is constant at k^*. The wage and rental price are $w = F_N(k^*, 1)$ and $r^* = F_K(k^*, 1) = \beta^{-1} - 1 + \delta$. The steady state consumption level in an economy without taxes is $c^i = w + (1-\beta)k^i$. In an economy with a tax z^i on individual i, we find $c^i = w + (\beta^{-1} - 1)k^i + (1-\beta)z^i$.

i. Since this is merely a redistribution of capital, the aggregate capital stock stays constant and the economy remains in the steady state. All aggregate variables are unchanged. Obviously, those taxed choose a lower level of consumption relative to those who receive the transfer. There may exist a tax and transfer scheme that generates $c^i = c^j$, $\forall\ (i, j)$. These allocations cannot be Pareto ranked with the original one.
More formally, we check that markets clear.

$$\begin{aligned}
\int_0^1 c^i di &= wN + (\beta^{-1} - 1)\int_0^1 k^i di + (1-\beta)\int_0^1 z^i di, \\
C &= F_N N + F_K K - \delta K + (1-\beta)\int_0^1 z^i di, \\
C &= F(K, N) - \delta K + (1-\beta)\int_0^1 z^i di.
\end{aligned}$$

In this question there is market clearing because $\int_0^1 z^i di = 0$.

ii. Same answer as in part *i.* because aggregate consumption does not change and neither does the aggregate capital stock. Again there is market clearing because $\int_0^1 z^i di = 0$.

iii. We consider two scenarios. In the first one $G_t = g = \int_0^1 z^i di\ \forall t$. The economy stays in the steady state. The steady state level of capital is unchanged (and likewise for the interest rate and the wage). The aggregate level of consumption is lower by the amount of government spending. The tax reduces the consumption of those taxed.

In a second scenario $G_0 = \int_0^1 z^i di > 0,\ G_t = 0\ \forall t \geq 1$. Then at $t = 1$, the market does not clear at the steady state capital level - consumption pair. The tax acts as an aggregate shock to the economy. The economy slowly returns to the consumption capital level steady state.