Problem Set 3
Answer Key

December 16, 2016

Problem 1. Endogenous Fertility
As we argued in class, population in our one-sector growth model is a fixed factor which is not accumulated and thus implies that there can be no growth in the long-run - if we have constant returns to scale production function. While this is a valid argument, perhaps the assumption that population is a fixed factor is not a good assumption. In particular, one can imagine that the decision to have children is one which is actually affected by economic conditions. Here we try to develop this insight and work on a model with endogenous fertility. This is based on the paper by Barro and Becker.

Consider the following intergenerational setting where in each period a new generation of individuals are born. They have the following preferences

$$W_t = u\left(c_t\right) + \beta \frac{g\left(n_t\right)}{n_t} \int_0^{n_t} W_{t+1}^i di$$

In other words, each generation lives for one period and get utility from their own consumption and the utility of their kids. The function $g\left(n\right)/n_t$ captures the degree of altruism and depends on the number of children. Note that we assumed that the number of children is continuous choice. At an aggregate level, this is not a crazy assumption. If we assume that all the children are the same, then we can write

$$W_t = u\left(c_t\right) + \beta \frac{g\left(n_t\right)}{n_t} \int_0^{n_t} W_{t+1}^i di = u\left(c_t\right) + \beta g\left(n_t\right) W_{t+1}$$

Thus combining all of this, we have

$$W_t = \sum_{s=t}^{\infty} \beta^{s-t} \left(g\left(n_t\right) \cdots g\left(n_{s-1}\right)\right) u\left(c_s\right)$$

The total population is given by

$$N_t = N_0 n_0 \cdots n_{t-1}$$

On the cost side, cost of children is in terms of time. In other words, if an individual decides to have n_t children, this will cost $b n_t w_t$ where w_t is the wage rate in the economy. In other words, every child will lead to a loss of b units of labor supply.

Assume that the individuals have access to capital and inelastically provide labor and that the production function is given by a standard Cobb-Douglass Production function using capital and labor.

A useful notation for this problem is to think about aggregate and per capita variables, i.e., $c_t = \frac{C_t}{N_t}$, $y_t = \frac{Y_t}{N_t}$, etc.

a. What is output in this economy given that population is given by N_t while capital is K_t? Derivate the feasibility constraint.

Solution. Output is given by the Cobb-Douglas production function which accounts for the loss of aggregate time for child care: $F(K_t, (1 - bn_t)N_t) = K_t^\alpha[(1 - bn_t)N_t]^{1-\alpha}$. The feasibility constraint is then given by

$$C_t + K_{t+1} \leq F(K_t, (1 - bn_t)N_t) + (1 - \delta)K_t$$

dividing each side with N_t we have the following per-capita feasibility constraint:

$$c_t + n_t k_{t+1} \leq k_t^\alpha (1 - bn_t)^{1-\alpha} + (1 - \delta)k_t$$

b. Write the budget constraint for a dynasty of households in this economy. Do this using a sequential market setting.

Solution. The budget constraint for a dynasty in terms is given by

$$p_{c,t}C_t + p_{x,t}X_t + q_t A_t = (1 - bn_t)N_t w_t + r_t K_t + p_{c,t}A_{t-1}$$

c. Define a competitive equilibrium for this economy. Note that since there is no heterogeneity, it is without loss of generality to assume that saving is done in the form of capital.

Solution. Competitive equilibrium is prices $\{p_{c,t}, p_{x,t}, w_t, r_t\}_{t=0}^\infty$ and allocation $\{C_t, X_t, N_t, K_t, K_t^f, L_t^f\}_{t=0}^\infty$ and fertility sequence $\{n_t\}_{t=0}^\infty$ such that

- Given prices the allocation and fertility choices solves the problem of the dynasty

$$\max_{C_t, X_t, K_{t+1}^h, n_t, , L_t^h} \sum_{t=s}^{\infty} \beta^{t-s}(g(n_s)...g(n_{t-1}))u(C_t)$$

  subject to the budget constraint

$$p_{c,t}\left(C_t + X_t\right) = \left(1 - bn_t\right) w_t N_t + r_t K_t,$$

  and the law of motion for capital and population

$$K_{t+1} = X_t + (1 - \delta)K_t$$

  with non-negativity constraint and population constraint on labor force.

- Given prices the allocation solves firms profit maximization problem

$$\max_{L_t^f, K_t} p_{c,t}F(K_t, (1 - bn_t)L_t^f) - w_t L_t^f - r_t K_t^f$$

- Markets clear

$$K_t^f = K_t$$
$$L_t^f = N_t\,(1 - bn_t)$$
$$C_t + X_t = F(K_t, (1 - bn_t)N_t)$$

d. Write the planning problem for this economy for a planner that maximizes the welfare of the initial generation. Show that the solution to this is part of a competitive equilibrium defined above. In other words, find prices for which any solution to the planning problem is a CE.

Solution. Social planners problem is

$$\max_{n_t, c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t g(n_0)...g(n_{t-1})u(\frac{C_t}{N_t})$$

subject to

$$C_t + K_{t+1} \leq K_t^{\alpha}\left(\left(1 - bn_t\right)N_t\right)^{1-\alpha} + (1-\delta)K_t$$
$$N_{t+1} = n_t N_t$$

In order to constructing prices, given a solution of the above problem, we can define

$$p_{c,t} = 1$$
$$w_t = F_N\left(K_t, \left(1 - bn_t\right)N_0 n_0 \cdots n_{t-1}\right)$$
$$r_t = F_K\left(K_t, \left(1 - bn_t\right)N_0 n_0 \cdots n_{t-1}\right)$$

Since production function is homogeneous of degree 1, given the above definitions, households budget constraint as described in part c is obviously satisfied. This implies that the allocation constitutes a competitive equilibrium since the problem is time-consistent.

e. Suppose that $u\left(c\right) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 1$, what do we have to assume about $g\left(n\right)$ so that this problem is mathematically well-behaved, i.e., fertility is non-zero?

Solution. Under the assumption, above utility have a child is negative - because $u\left(c\right) < 0$. In order for households to have any children, we must have that $g'\left(n\right) < 0$, otherwise no will have any children. If we further assume that $g'\left(0\right) = -\infty$, this will ensure that number of children is positive.

f. Write the planning problem recursively. What are the state variables?

Solution. The Bellman equation is given by

$$v\left(k, N\right) = \max_{k', n} u\left(k^{\alpha}\left(1 - bn\right)^{1-\alpha} + (1-\delta)k - nk'\right) + g\left(n\right)\beta v\left(k', Nn\right)$$

The state variable in this formulation is the current population and capital holdings. We can further simplify the problem and reduce the state space to k by examining the above and noticing that the problem is independent of N. In other words, changing the level of population does not change the per-capita variables - only aggregates.

g. Suppose that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$ and that $g(n) = n^{1+\eta}$ what assumption on η and σ should be made so that people will always have children?

Solution. If we write the objective function in terms of N_t, we have

$$\sum_{t=0}^{\infty} \beta^t N_t^{1+\eta} \frac{(C_t/N_t)^{1-\sigma}}{1-\sigma}$$

For the problem to be well-defined, we need to have $\eta + \sigma \leq 0$.

h. Assume the above specification, define a BGP for this economy. Suppose a BGP exists. What is the long-run growth rate of the economy?

Solution. A BGP in this economy is where all variables, C_t, K_t, N_t grow at a constant rate which by definition will be equal to population growth n.

Two Euler equations determine the dynamic trade-offs in this model. If we write down the problem in terms of the aggregates, we have

$$\max \sum_{t=0}^{\infty} \beta^t N_t^{\eta+\sigma} \frac{C_t^{1-\sigma}}{1-\sigma}$$

subject to

$$C_t + K_{t+1} = K_t^{\alpha} (N_t - bN_{t+1})^{1-\alpha} + (1-\delta) K_{t+1}$$

The Euler equation w.r.t. K_{t+1} is

$$N_t^{\eta+\sigma} C_t^{-\sigma} = \beta N_{t+1}^{\eta+\sigma} C_{t+1}^{-\sigma} [1 - \delta + F_{K,t+1}]$$

The one w.r.t N_{t+1} is

$$bF_{N,t} N_t^{\eta+\sigma} C_t^{-\sigma} = \beta (\eta + \sigma) N_{t+1}^{\eta+\sigma-1} \frac{C_{t+1}^{1-\sigma}}{1-\sigma} + \beta N_{t+1}^{\eta+\sigma} C_{t+1}^{-\sigma} F_{N,t+1}$$

where the first term on the RHS captures the increase from utility of having more kids while the second term captures the increase in the labor force and aggregate labor income. We thus have that in a BGP

$$1 = \beta n^{\eta} [1 - \delta + F_K] \tag{1}$$

$$bF_N = \beta \frac{\eta + \sigma}{1-\sigma} n^{\eta} c + \beta n^{\eta} F_N \tag{2}$$

$$c + nk = k^{\alpha} (1 - bn)^{1-\alpha} + (1-\delta) k$$

The above is a system of equations in three unknowns, c, n, k and its solution pins down the long-run growth in a BGP.

i. Suppose that $-\sigma = \eta$, calculate the long-run growth rate of the economy. Which of the models that we have discussed in class is similar to this? Answer intuitively or mathematically. **Note: There was a typo in the statement of the problem.**

Solution. In this case, equation 2 becomes

$$b = \beta n^{\eta}$$

which means that the growth in the economy is given by

$$(b/\beta)^{1/\eta}$$

If we rewrite the problems in terms of the aggregates, we have

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

subject to

$$C_t + K_{t+1} = K_t^{\alpha} \left(N_t - bN_{t+1}\right)^{1-\alpha} + (1-\delta) K_{t+1}$$

This is basically, a model with two types of capital: population and physical capital. It thus is very similar to the AKH model.

j. For the economy in part i, calculate the population growth rate. How is this correlated with the growth rate of the economy. Discuss this relationship and relate it to what we observe in the data. Are they in line? If not, what is the main issue with the model developed above.

Solution. Population growth rate in the long-run is basically equal to the growth rate of aggregate variables; note that as usual growth rate of per-capita variables are zero. There are two ways to think about the data:

1. data is really coming from this model as we transit to steady state where aggregate capital stock increases. Under this interpretation, the Euler equation for capital and the fact that $\eta < 0$, implies that as capital stock increases, population growth decline. This is in line with the observed data on population and growth. This view, however, is problematic as we have discussed in class since we know in the data growth typically coincides with periods of growth in productivity.

2. We can add exogenous productivity growth. Under this, the Euler equation for population is given by

$$b = \beta \frac{n^{\eta}}{g^{\sigma}} \Rightarrow n = (b/\beta)^{\frac{1}{\eta}} g$$

Under this specification there is a positive relationship between fertility and growth. This mainly comes from the wealth effect on fertility - the richer a country, the more kids they want.

Problem 2. Growth with Externality and Taxes

Consider Romer's model of growth with externality that we have discussed in class.

a. Define a Pareto optimal allocation.

Solution. A Pareto optimal allocation in this economy is the solution of the following social planner's problem

$$\max_{c_t, k_{t+1}} \sum_{t=0} \beta^t u(C_t)$$

subject to

$$C_t + K_{t+1} \leq F(K_t, AK_tL) + (1-\delta)K_t$$

b. Formulate a social planning problem associated with Pareto optimality and find its solution.

Solution. Social planner's problem is given by

$$\max_{c_t,k_{t+1}} \sum_{t=0} \beta^t u(C_t)$$

subject to

$$C_t + K_{t+1} \leq F(K_t, AK_tL) + (1-\delta)K_t$$

Here ,it is important to note that Pareto problem internalizes the spill-over effects as the social planner takes into account of the effect of current capital stock on the labor productivity. Also I suppress the subscript on L as labor is supplied inelastically in this economy. Given that u is strictly increasing, we can substitute the constraint back into problem

$$\max_{c_t,k_{t+1}} \sum_{t=0} \beta^t u(F(K_t, AK_tL) + (1-\delta)K_t - K_{t+1})$$

Taking the first order condition w.r.t K_{t+1} we have the Euler equation

$$u'(C_t) = \beta u'(C_{t+1})[F_1(K_{t+1}, AK_{t+1}L) + AL_tF_2(K_{t+1}, AK_{t+1}L) + (1-\delta)]$$

where $F_1(K_t, AK_tL_t) + AL_tF_2(K_t, AK_tL_t)$ is the total derivative of F with respect to K_{t+1}.

c. Define a TDCE for this economy assuming that there is a tax or subsidy on capital income only.

Solution. A Tax Distorted Competitive Equilibrium for this economy is an allocation $\{C_t, X_t, K_t^h, K_t^f, L_t^H, L_t^f\}$, prices $\{p_t, w_t, r_t\}$, and a policy plan $\{\tau_{t,k}, T_t\}$ such that

- Given prices the allocations solve the households problem

$$\max_{C_t,X_t,K_t^h,L_t^h} \sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t[C_t + X_t] \leq \sum_{t=0}^{\infty}(1-\tau_{t,k})r_tK_t^h + w_tL_t^h + p_tT_t$$
$$K_{t+1} = (1-\delta)K_t + X_t$$

- Given prices the allocations solve the firms problem

$$\max_{K_t^f,L_t^f} p_tF(K_t^f, BL_t^f) - r_tK_t^f - w_tL_t^f$$

Note that K^f is the capital requirement for an infinitesimal firm (even though we are using a representative firm here) and therefore has no effect on the $B = AK_t$ where the K_t is the aggregate capital in the economy.

- Markets clear & governments budget constraint is satisfied

$$K_t^h = K_t^f$$
$$L_t^h = L_t^f$$
$$C_t + X_t = F(K_t^f, BL_t^f)$$
$$\sum_{t=0}^{\infty} \tau_{t,k} r_t K_t^h = \sum_{t=0}^{\infty} p_t T_t$$

d. Find a tax rate so that the TDCE above coincides with the pareto optimal allocation that you found in part b. Is this a tax or subsidy? Provide an intuition for your answer.

Solution. The first order conditions to the households problem given above are

$$\beta^t u'(C_t) = \lambda p_t$$
$$\beta^{t+1} u'(C_{t+1}) = \lambda p_{t+1}$$
$$p_t = [(1 - \tau_{t+1,k}) r_{t+1} + (1 - \delta) p_{t+1}]$$

The first order condition to firms problem with respect to capital gives

$$p_t F_1(K_t, BL_t) = r_t$$

Iterating firms foc one period and solving these four equation together result in the Euler Equation

$$u'(C_t) = \beta u'(C_{t+1})[(1 - \tau_{t+1,k}) F_1((K_{t+1}, BL_{t+1}) + (1 - \delta)]$$

Comparing this Euler Equation with the Euler equation from the planners problem we observe that for Pareto optimality we need a specific stream of capital taxes $\{\tau_{t,k}\}$ given by

$$\tau_{t,k} = 1 - \frac{F_1(K_t, AK_t L) + AL_t F_2(K_t, AK_t L)}{F_1((K_t, AK_t L)}$$

so that the Euler Equations are identical. Since the positive spill-over effects are increasing with aggregate capital; we have $AL_t F_2(K_t, AK_t L) > 0$ and $F_1(K_t, AK_t L) > 0$ as production function is an increasing function of capital. These together implies that the $\tau_{t,k}$ is negative. Hence to attain Pareto optimality we need to subsidize capital holdings.

Intuitively what is going on is, as individual firms do not internalize the positive externalities, the interest rate on capital is priced below the socially optimum level. Therefore, the households under-save. By subsidizing capital we are incentivizing them to save at the socially optimum level and hence attain the Pareto optimal allocation.

Problem 3. Equivalent Tax Systems

Solve Exercise 16.7-16.9 in Ljungqivst and Sargent.

Problem 4. Optimal Taxes in the AKH model

Consider the AKH model (with cost of human capital accumulation in terms of the final good) that we discussed in class and assume that the government can impose taxes/subsidies on labor income, capital income and consumption of households. Recall that production function was Cobb-Douglas, preferences are given by $\sum_{t=0}^{\infty} \beta^t C_t^{1-\sigma} / (1-\sigma)$ and that depreciation of physical and human capital are the same.

a. Define a TDCE in this economy.

Solution. A Tax Distorted Competitive Equilibrium for this economy is an allocation $\{C_t, Z_t, H_t^h, H_t^f X_t, K_t^h, K_t^f$ prices $\{p_t, w_t, r_t\}$, and a policy plan $\{\tau_{t,c}, \tau_{t,l}, \tau_{t,k}, \tau_{x,t}, \tau_{z,t}, G_t\}$ such that

- Given prices the allocations solve the households problem

$$\max \sum_{t=0}^{\infty} \beta^t C_t^{1-\sigma} / (1-\sigma)$$

subject to

$$\sum_{t=0}^{\infty} p_t[(1+\tau_{t,c})C_t + (1+\tau_{z,t}) Z_t + (1+\tau_{x,t}) X_t] \leq \sum_{t=0}^{\infty} (1-\tau_{t,k}) r_t K_t^h + (1-\tau_{t,l}) w_t H_t^h$$
$$K_{t+1} = (1-\delta)K_t + X_t$$
$$H_{t+1} = (1-\delta)H_t + Z_t$$

- Given prices the allocations solve the firms problem

$$\max_{K_t^f, H_t^f} p_t A K_t^\alpha H_t^{1-\alpha} - r_t K_t^f - w_t H_t^f$$

- Markets clear and governments budget constraint is satisfied

$$K_t^h = K_t^f$$
$$H_t^h = H_t^f$$
$$C_t + Z_t + X_t + T_t = A K_t^\alpha H_t^{1-\alpha}$$
$$\sum_{t=0}^{\infty} [\tau_{c,t} p_t C_t + \tau_{k,t} r_t K_t + \tau_{l,t} w_t H_t + \tau_{x,t} p_t X_t + \tau_{z,t} Z_t] = \sum_{t=0}^{\infty} p_t G_t$$

b. Show that an allocation is part of a TDCE if and only if it is feasible and satisfies an implementability constraint - derive the implementability constraint.

Solution. The first order conditions of households problem and firms problem are given by

$$\begin{aligned}
C_t :&\ \beta^t C_t^{-\sigma} = \lambda p_t (1+\tau_{c,t}) \\
K_{t+1} :&\ \lambda p_t = \lambda[(1-\tau_{k,t+1}) r_{t+1} + p_{t+1}(1-\delta)] \\
H_{t+1} :&\ \lambda p_t = \lambda[(1-\tau_{l,t+1}) w_{t+1} + p_{t+1}(1-\delta)] \\
K_t :&\ r_t = \alpha A K_t^{\alpha-1} H_t^{1-\alpha} \\
H_t :&\ w_t = (1-\alpha) A K_t^{\alpha} H_t^{-\alpha}
\end{aligned}$$

To derive the implementability condition we will use the first order conditions to manipulate the households budget constraint (given that the utility is strictly increasing)

$$\sum_{t=0}^{\infty} p_t[(1+\tau_{t,c})C_t + Z_t + X_t] = \sum_{t=0}^{\infty}(1-\tau_{t,k})r_t K_t + (1-\tau_{t,l})w_t H_t$$

$$\sum_{t=0}^{\infty}(1+\tau_{t,c})p_t C_t + p_t\left(1+\tau_{z,t}\right)Z_t - (1-\tau_{t,l})w_t H_t = \sum_{t=0}^{\infty}(1-\tau_{t,k})r_t K_t + \left(1-\tau_{x,t}\right)p_t(1-\delta)K_t - \left(1-\tau_{x,t}\right)p_t$$

$$\sum_{t=0}^{\infty}(1+\tau_{t,c})p_t C_t + \left(1+\tau_{z,t}\right)p_t Z_t - (1-\tau_{t,l})w_t H_t = (1-\tau_{k,0})r_0 K_0 + p_0(1-\delta)K_0 + \sum_{t=0}^{\infty}(1-\tau_{k,t+1})r_{t+1}K_{t+}$$

The summation at the l.h.s. is zero as a result of the first order condition w.r.t. k_{t+1} and transversality condition $\lim_{t\to\infty} p_t k_{t+1} = 0$. Doing the same manipulations to human capital as we did for physical capital results in

$$\sum_{t=0}^{\infty}(1+\tau_{t,c})p_t C_t = (1-\tau_{k,0})r_0 K_0 + p_0(1-\delta)K_0 + (1-\tau_{l,0})w_0 H_0 + p_0(1-\delta)H_0$$

From the first order condition for consumption, normalizing $p_0 = 1$ we get $p_t(1+\tau_{c,t}) = \beta^t \frac{u'(Ct)}{u'(C_0)}$. Substituting this expression we arrive at the implementability condition:

$$\sum_{t=0}^{\infty}\beta^t C_t^{-\sigma} C_t = C_0^{-\sigma}[(1-\tau_{k,0})r_0 K_0 + p_0(1-\delta)K_0 + (1-\tau_{l,0})w_0 H_0 + p_0(1-\delta)H_0]$$

Proposition. *An allocation is part of a TDCE if and only if it is feasible and satisfies an implementability constraint.*

Proof. TDCE implies Feasibility and Implementability: Given an allocations is TDCE feasibility holds as markets clear. Implementability condition holds first order conditions and budget constraint is satisfied as derived above.

Feasibility + Implementability implies TDCE: Given an allocation we can construct the prices and the taxes using the first order conditions. Under these prices the households problem is solved as given these prices the allocation respects optimality conditions and since the implementability constraint holds the budget constraint is satisfied as we have derived above. Similarly firms problem is solved as the allocation also satisfies firms optimality as r_t and w_t respects firms optimality. Given feasibility and implementability holds governments budget constraint is satisfied by Walras' Law. □

c. Are there any redundant taxes in this model?

Solution. The Euler Equations for the household is given by

$$\begin{aligned}
\left(1+\tau_{x,t}\right)\left(1+\tau_{c,t}\right)C_t^{-\sigma} &= \beta(1+\tau_{c,t+1})C_{t+1}^{-\sigma}[(1-\tau_{k,t+1})r_{t+1} + (1-\delta)] \\
\left(1+\tau_{z,t}\right)\left(1+\tau_{c,t}\right)C_t^{-\sigma} &= \beta(1+\tau_{c,t+1})C_{t+1}^{-\sigma}[(1-\tau_{l,t+1})w_{t+1} + (1-\delta)]
\end{aligned}$$

Hence we have two taxes to pin down using one equation. This implies that of the five tax rates, three are redundant.

d. Calculate optimal capital and labor income taxes.

Solution. Formulate the Ramsey problem as

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$

subject to

$$C_t + G_t + K_{t+1} + H_{t+1} = AK_t^{\alpha} H_t^{1-\alpha} + (1-\delta)(K_t + H_t)$$
$$\sum_{t=0}^{\infty} \beta^t C_t^{1-\sigma} = C_0^{-\sigma}[(1-\tau_{k,0})r_0 K_0 + p_0(1-\delta)K_0 + (1-\tau_{l,0})w_0 H_0 + p_0(1-\delta)H_0]$$

Let λ_t and μ be the Lagrange multipliers on feasibility and implementability constraints respectively. First order conditions are

$$\begin{aligned}
\lambda_t &= (1-\mu(1-\sigma))\beta^t C_t^{-\sigma} \quad t \geq 1 \\
\lambda_t &= \lambda_{t+1}(\alpha A K_t^{\alpha-1} H_t^{1-\alpha} + (1-\delta)) \\
\lambda_t &= \lambda_{t+1}((1-\alpha)AK_t^{\alpha} H_t^{-\alpha} + (1-\delta))
\end{aligned}$$

Hence at the optimality we have

$$\begin{aligned}
\alpha A K_t^{\alpha-1} H_t^{1-\alpha} &= (1-\alpha) A K_t^{\alpha} H_t^{-\alpha} \\
\alpha \left[\frac{K}{H}\right]_t^{\alpha-1} &= (1-\alpha)\left[\frac{K}{H}\right]_t^{\alpha} \\
\left[\frac{K}{H}\right]_t &= \frac{\alpha}{(1-\alpha)}
\end{aligned}$$

On the other hand from our previous analysis we had

$$\begin{aligned}
(1-\tau_{k,t})r_t &= (1-\tau_{l,t})w_t \\
(1-\tau_{k,t})\alpha A K_t^{\alpha-1} H_t^{1-\alpha} &= (1-\tau_{l,t})(1-\alpha) A K_t^{\alpha} H_t^{-\alpha} \\
\left[\frac{K}{H}\right]_t &= \frac{(1-\tau_{k,t})}{(1-\tau_{l,t})} \frac{\alpha}{(1-\alpha)}
\end{aligned}$$

Therefore optimality requires taxes to be equal on labor and capital (if there is any). The Euler Equation is given by

$$\begin{aligned}
C_t^{-\sigma} &= \beta C_{t+1}^{-\sigma}[\alpha A K_t^{\alpha-1} H_t^{1-\alpha} + (1-\delta)] \\
C_t^{-\sigma} &= \beta C_{t+1}^{-\sigma}[\alpha A \left[\frac{K}{H}\right]_t^{\alpha-1} + (1-\delta)] \\
C_t^{-\sigma} &= \beta C_{t+1}^{-\sigma}[\alpha A \left[\frac{\alpha}{1-\alpha}\right]^{\alpha-1} + (1-\delta)]
\end{aligned}$$

Assuming there exists a balanced growth path for this economy in the long run we have

$$\beta[\alpha A \left[\frac{\alpha}{1-\alpha}\right]^{\alpha-1} + (1-\delta)] = 1+g$$

This equation is important for two reasons: First the growth rate in this economy is a function of the parameters. Therefore there is constant (endogenous) growth. Moreover, the transition happens almost immediately as the optimal K/H ratio will be attained in one period. Now let's compare this results to the TDCE results from above. The Euler equation (for capital) was given by

$$\begin{aligned}
C_t^{-\sigma} &= \beta C_{t+1}^{-\sigma}[(1-\tau_{k,t+1})r_{t+1} + (1-\delta)] \\
C_t^{-\sigma} &= \beta C_{t+1}^{-\sigma}[(1-\tau_{k,t+1})\alpha A K_t^{\alpha-1} H_t^{1-\alpha} + (1-\delta)] \\
C_t^{-\sigma} &= \beta C_{t+1}^{-\sigma}[(1-\tau_{k,t+1})\alpha A K_t^{\alpha-1} H_t^{1-\alpha} + (1-\delta)] \\
C_t^{-\sigma} &= \beta C_{t+1}^{-\sigma}[(1-\tau_{k,t+1})\alpha A \left[\frac{(1-\tau_{k,t})}{(1-\tau_{l,t})}\frac{\alpha}{(1-\alpha)}\right]^{\alpha-1} + (1-\delta)]
\end{aligned}$$

Using the fact that at optimality we have $\tau_{k,t} = \tau_{l,t}$

$$C_t^{-\sigma} = \beta C_{t+1}^{-\sigma}[(1-\tau_{k,t+1})\alpha A \left[\frac{\alpha}{(1-\alpha)}\right]^{\alpha-1} + (1-\delta)]$$

Then at the BGP

$$\beta[(1-\tau_{k,t+1})\alpha A \left[\frac{\alpha}{(1-\alpha)}\right]^{\alpha-1} + (1-\delta)] = 1+g$$

Therefore we have the zero-capital tax on the balanced growth path. However, here there is more to this limiting result. As per discussed in the lecture notes this economy converges to its BGP quite fast, and the transition will occur in one period. Therefore we have $\tau_{k,t} = 0$ for $t \geq 1$. Which implies $\tau_{l,t} = 0$ for $t \geq 1$ as well.

Problem 5. Uniform Commodity Taxation with Two Goods - Non-Primal approach Suppose that we have an economy where there are two consumption goods and individuals have some wealth of the numeraire good. In addition, suppose that prices of both goods are given by $p_i = 1, i = 1, 2$.

a. Formulate the optimal taxation problem.

Solution. The households problem is

$$\max u(c_1, c_2)$$

subject to

$$(1+\tau_1)c_1 + (1+\tau_2)pc_2 = \omega + w$$

The first order conditions then is

$$\begin{aligned}
u_1 &= \lambda(1+\tau_1) \\
u_2 &= \lambda(1+\tau_2)p
\end{aligned}$$

Substituting the first order conditions back into budget constraint we have following implementability condition

$$u_1 c_1 + u_2 c_2 = \lambda(\omega + w)$$

Firms in this economy maximizing their profits

$$\max p_i n_i - w n_i$$

The first order condition is

$$p_i = w = 1$$

Then the Ramsey problem is given by

$$\max u(c_1, c_2)$$

subject to feasibility and implementability

$$\sum_{i=1,2} c_i + g_i = 1 + \omega$$

$$u_1 c_1 + u_2 c_2 = \lambda(\omega + 1)$$

b. Show that if utility of the individuals in the economy are given by $u(c_1, c_2) = \alpha_1 \frac{c_1^{1-\sigma}}{1-\sigma} + \alpha_2 \frac{c_2^{1-\sigma}}{1-\sigma}$, then optimal taxes over the two goods are the same.

Solution. Let μ_i refer to the Lagrange multipliers on the respective constraints. The first order conditions of the Ramsey problem is

$$\alpha_1 c_1^{-\sigma} - \mu_2 \alpha_1 (1-\sigma) c_1^{-\sigma} = \mu_1$$
$$\alpha_2 c_2^{-\sigma} - \mu_2 \alpha_2 (1-\sigma) c_2^{-\sigma} = \mu_1$$

which implies at the optimal allocation we have

$$\alpha_1 c_1^{-\sigma} = \alpha_2 c_2^{-\sigma}$$

plugging this to the first order condition of the household

$$\frac{\alpha_1 c_1^{-\sigma}}{(1+\tau_1)} = \frac{\alpha_2 c_2^{-\sigma}}{(1+\tau_2)}$$
$$\tau_1 = \tau_2$$

c. bonus question: What happens to the optimal taxes when individuals are heterogeneous with respect to their wealth of the numeraire?

Solution. With the assumption of CRRA utility function the demand functions do not exhibit income effects. Therefore, the consumption choices of different individuals are the same proportionately. Therefore, in this economy wealth heterogeneity would not change the uniform taxation result.