1 Search and Unemployment

1.1 McCall's Model of Sequential Search

So far, in our models, we have discussed employment but not unemployment. This is mainly because of the intricate definition of unemployment. It is official defined as the fraction of the labor force that does not have a job where labor force is defined as people who either have a job or are actively looking for one. One way to think about unemployment is as stemming from search frictions: workers cannot necessarily find all the jobs and they might find a job that they do not like.

The McCall search model is a simple partial equilibrium model that captures this idea. Suppose that the economy is populated by a continuum of workers and a continuum of jobs. Jobs have wages attached to them and are assumed to be heterogeneous in the wage they offer. Suppose that wages are distributed according to a c.d.f. function $F(w)$ with $Supp(F) = [\underline{w}, \overline{w}]$ and that if a job gets taken by a worker, there will be another with the same wage and therefore, unemployed workers always face the same distribution of wage offers. We assume that once workers accept a job offer, they will work at this job forever. We assume that workers are risk neutral; cannot save and discount the future at rate β. Moreover, they find a job with probability λ. Also, we assume that unemployed workers receive a benefit b when they are unemployed; this could be a stand-in for unemployment benefits, value of home production, leisure, etc.

The easiest way to analyze this problem is by working directly with value functions. Let $V(w)$ be value of being employed at a job that pays w and let U be the value of being unemployed. Then

$$\begin{aligned} U &= b + \beta\lambda \int \max\{V(w), U\}\, dF(w) + \beta(1-\lambda) U \\ V(w) &= \frac{w}{1-\beta} \end{aligned}$$

The first equation captures the idea that value of unemployment is equal to its flow value plus a continuation value. When an unemployed worker finds a job - with probability λ - she can decide whether to accept this job and get a value of $V(w)$ or to reject and get value associated with unemployment, U. With a complimentary probability, she will not find a job in which case her value is simply U. The second equation simply states that the value of employment at a job with wage w is $\frac{w}{1-\beta}$ the value of earning w forever. We can write the first equation as

$$U - \beta U = b + \beta\lambda \int \max\left\{\frac{w}{1-\beta} - U, 0\right\} dF(w)$$

Thus a reservation wage w^* exists for which the unemployed accept all offers above w^* and reject all offer below. Furthermore, we must have $U = \frac{w^*}{1-\beta}$. We have

$$w^* = b + \beta\lambda \int_{w^*}^{\overline{w}} \frac{w - w^*}{1-\beta} dF$$
$$w^* = b + \frac{\beta}{1-\beta}\lambda \int_{w^*}^{\overline{w}} (w - w^*)\, dF$$

This equation is intuitive. It states that the reservation wage is equal to unemployment benefit plus the option value of waiting for a new offer as captured by the second term in the RHS. The RHS is decreasing in w^*, while the LHS is increasing. Therefore, a unique solution to above exists. If we assume that $\overline{w} > b$, then the solution to the above is higher than b and lower than $\overline{w}$. In addition, it can be shown that an increase in b leads to an increase in w^*. To see this, we have

$$dw^* = db + \frac{\beta}{1-\beta}\lambda \int_{w^*}^{\overline{w}} (-dw^*)\, dF$$
$$\left[1 + \frac{\beta\lambda}{1-\beta}\left(1 - F(w^*)\right)\right] dw^* = db$$
$$\frac{dw^*}{db} = \frac{1}{1 + \frac{\beta\lambda}{1-\beta}\left(1 - F(w^*)\right)}$$

So this model captures the idea that an increase in unemployment benefits makes the workers more picky with respect to jobs and leads to higher unemployment.

The McCall search model is very elegant and simple yet it is potentially rich enough to consider various extensions and modifications. For example, one can easily think about exogenous separation in this model. That is suppose that jobs terminate with probability η in each period. Then, the value of unemployed is the same as before. However, the value of being employed is not given by

$$V(w) = w + \beta(1-\eta)V(w) + \beta\eta U \to V(w) = \frac{w + \beta\eta U}{1 - \beta(1-\eta)}$$

so now the two values are interdependent. The simplicity of the model also allows us to examine various moments. For example, we can think about measure of wage dispersion in this model - since wage dispersion is endogenous. One statistic related to that is the min-mean ratio: the ratio of minimum observed wage to average wage

$$\frac{w^*}{w_m} = \frac{w^*}{\mathbb{E}\left[w|w \geq w^*\right]} = \frac{b + \frac{\beta}{1-\beta}\lambda\left[1 - F\left(w^*\right)\right]\left(w_m - w^*\right)}{w_m}$$
$$\frac{w^*}{w_m} = \frac{\frac{b}{w_m} + \frac{\beta}{1-\beta}\lambda\left[1 - F\left(w^*\right)\right]}{1 + \frac{\beta}{1-\beta}\lambda\left[1 - F\left(w^*\right)\right]}$$

In the above formula, $\lambda\left(1 - F\left(w^*\right)\right)$ is the rate at which the unemployed find jobs and thus can be measured directly in the data. A paper by Hornstein et al. (2011) does the measurement on the RHS and shows that the model cannot explain the min-mean ratio observed in the data.

Sometimes, in search models, it is useful to formulate the problem in continuous time. This is because some Bellman equations are easier to analyze in continuous time. To do this, I consider the above model and will try to send the time interval to zero. In particular, suppose that the length of the time interval is Δ. Furthermore, suppose that the probability of finding a job is given by $\eta\Delta$ and the discount factor is given by

$$\frac{1}{1 + \rho\Delta}$$

Note that as we send Δ to zero, the discount factor converges to 1 and the probability of finding a job converges to 0. This makes sense as the probability of finding a job in the next instant is probably 0. With continuous time, the appropriate way to think about η and ρ is rate at which the probability of finding a job increases with time and the rate at which the individual discounts the future, respectively. Note also that as we send Δ to zero, we should also decrease wages and unemployment benefit. As in the case of job finding rate, we can talk about wage rate and benefit rate. In other words, for any period of length Δ, $b\Delta$ is the unemployment benefit received by the individual during the period and $w\Delta$ is the wage received by a worker. Therefore, we have

$$V_\Delta\left(w\right) = \frac{w\Delta}{1 - \frac{1}{1+\rho\Delta}} = \frac{w\left(1 + \rho\Delta\right)}{\rho}$$
$$\left(1 - \frac{1}{1 + \rho\Delta}\right)U_\Delta = b\Delta + \frac{\eta\Delta}{1 + \rho\Delta}\int \max\left\{V_\Delta\left(w\right) - U_\Delta, 0\right\}dF\left(w\right)$$
$$\frac{\rho}{1 + \rho\Delta}U_\Delta = b + \frac{\eta}{1 + \rho\Delta}\int \max\left\{\frac{w\left(1 + \rho\Delta\right)}{\rho} - U_\Delta, 0\right\}dF\left(w\right)$$

Therefore, as Δ converges to 0, the limit of the above are given by

$$V(w) = \frac{w}{\rho}$$
$$\rho U = b + \eta \int \max\left\{\frac{w}{\rho} - U, 0\right\} dF(w)$$

The above value functions can be explained in words. The LHS is the decline in the continuation value of unemployed from one period to the next (infinitesimally next!):

$$\frac{d}{dt}\left(U - e^{-\rho t}U\right)\Big|_{t=0} = \rho U$$

The right hand side is the increase in the value of being unemployed. The unemployed person, collects benefits for sure and at rate η will find a job that she accepts if its value is above U. If w^* is the reservation wage, then $U = \frac{w^*}{\rho}$ and

$$w^* = b + \frac{\eta}{\rho}\int_{w^*}^{\overline{w}} (w - w^*)\, dF(w)$$

We can write the above as

$$\rho w^* = \rho b + \eta \int_{w^*}^{\overline{w}} (w - w^*)\, dF(w)$$

If we send ρ to 0, the above implies that w^* converges to $\overline{w}$. In other words, if the individual does not discount the future, then she becomes extremely picky and will only accept the best wage offer out there.

It is also easy to calculate various moments related to unemployment in this model. For example, the rate at which an unemployed person finds a job is given by

$$H = \eta\left[1 - F(w^*)\right]$$

To see this note that unemployment must satisfy

$$\dot{u}_t = -\eta\left[1 - F(w^*)\right] u_t$$

because in each period, an unemployed person finds a job at rate η and will accept it with probability $1 - F(w^*)$. That is the probability of a person who is unemployed at time 0 has found a job by t is $1 - e^{-Ht}$ whose density is given by He^{-Ht}. Therefore, average unemployment duration is given by

$$\int_0^\infty tHe^{-Ht}dt = -\int_0^\infty t\, d\left(e^{-Ht}\right) = \int_0^\infty e^{-Ht}dt = \frac{1}{H}$$

Thus average duration of unemployment is $\frac{1}{\eta[1-F(w^*)]}$. This would have been harder to calculate in discrete time.

The Rothschild Critique and Diamond Paradox

In the above example, an important assumption is that wages are exogenously given. The question is can we get this to come out of optimal behavior by firms. As it is suggested by the title, the result is negative. In fact, Rothschild critique and Dimaond paradox illustrate that it is difficult to get a wage distribution to come out of optimizing behavior by firms. To see this, suppose that firms are heterogeneous with respect to their productivity, z; they are capacity-constrained in that they only have one job and post their wage at time 0 and commit to this posted wage. Taking as given the reservation wage policy of the individual, it is clear that no firm should offer something higher than w^*. To see this suppose that some firm is posting a wage above the reservation wage w^*, then this firm can cut its wage by $\varepsilon > 0$ and small; workers will still accept the job offer since this lower wage is still above the reservation wage and thus the profits for the firm will go up. In other words, there wont be an equilibrium distribution of wages. Diamond (1971) went one step further. He said, suppose that all firms are posting a wage $w^* > b$. Then as long as $\beta < 1$, one firm has an incentive to deviate and post something lower than w^*. To see this, suppose that one firm posts a price $w^* - \varepsilon$. The value of accepting this offer for an unemployed worker is

$$\frac{w^* - \varepsilon}{1 - \beta}$$

while if she waits, her payoff is, at best, given by

$$U = b + \beta\eta\frac{w^*}{1-\beta} + \beta\left(1-\eta\right)U$$

which is less than $\frac{w^*}{1-\beta}$. Therefor, for $\varepsilon > 0$ and small enough, the value of accepting the offer $w^* - \varepsilon$ is higher than the value of waiting. This implies that in equilibrium, $w^* = b$. In other words, firms will fully exploit the workers and offer them the value of unemployment benefit and workers will accept any job offer right away. In other words, they are always indifferent between working and not working. This is puzzling because presumably when a firm is setting a wage today, they should be thinking that the worker always has the option to wait. But the existence of search cost makes a firm who has met the worker effectively a monopolist.

We will talk about two models that will resolve this issue. First, a model where some workers have two offers in their hands - thereby creating some sort of competition within a period among firms. Second, a model where firms do not commit to wages - no wage posting - and wages are determined as a result of a bargaining process.

1.2 The Burdett-Judd Model

The first idea was explored by Burdett and Judd (1983). To see a very simple version, suppose that now the economy is static. There are two firms that are identical and they post wages. There is one worker! The firms send out fliers and the worker might receive these fliers. The probability that the worker receives the flier of each firm is given by π. Therefore, there will be four possibilities for the worker: receive two fliers, receive a flier from firm 1 (or 2) and receive no fliers. The rest is the same as before: value of unemployment is b. Additionally, value of firm productivity is z and $z > b$ so it is always efficient for the worker to work for the firm.

Suppose, for now, that the firm can tell if a worker has both fliers or only one. Then the outcome is clear: when the worker is in contact with one firm, the firm basically offers her b and she will accept - basically the firm is a monopolist in this case. If the worker has two offers, then the firm knows that it is in competition with another firm and this competition leads to a wage equal to z. Again, we wont have a non-trivial distribution of wages.

Now, suppose that firms cannot tell whether a worker that they has their flier has received a flier from the other firm or not. Then they have to offer a single wage. This wage cannot be equal to their productivity, z, this is because they can always guarantee for themselves an expected profit of $(1-\pi)(z-b)$ - by basically offering b and only having a captive worker work for them. At the same time, in any symmetric equilibrium, if a firm offers any wage below z, its profits are given by

$$\left(\pi(1-\pi)+\frac{\pi^2}{2}\right)(z-w) \tag{1}$$

That is, it hires a non-captive worker with probability $1/2$ and hires a captive worker for sure. Given this, the firm has an incentive to increase the wage by ε and trading with a captive worker for sure. Its expected profits are given by

$$z-w-\varepsilon$$

For $\varepsilon > 0$ and small enough, the above value is higher than (1). This means that this game between the firms has no pure strategy symmetric equilibria. It, therefore, must have a mixed strategy equilibrium. Let the c.d.f. of the wage distribution used by each firm be given by $F(w)$. Then the payoff for a firm of offering any wage w is given by

$$\pi(1-\pi+\pi F(w))(z-w)$$

if $F(\cdot)$ does not have a mass point at w. That is, the firm will always hires a captive worker but is only able to hire a non-captive worker with probability $F(w)$ which is the probability that the other offer at hand is below w . Note that, if $F(w)$ has a mass point at w, i.e., a discontinuity, the payoff of the firm is

$$\pi\left(1-\pi+\pi F^{-}(w)+\frac{\pi}{2}\left(F(w)-F^{-}(w)\right)\right)(z-w)$$

where $F^{-}(w)$ is the left limit of F at w. The above states that if $F(w)$ has a mass point at w, then a firm offering w will get the non-captive worker who has an offer at hand of w with probability $\frac{1}{2}$.

It can be shown that $F(w)$ is nice looking! That is it has no holes and no mass point. To see this, suppose that $F(w)$ has a mass point at w which means the firm get a non-captive worker that has an offer w at hand with probability $1/2$. Now an increase in wages by ε causes the firm to attract the non-captive worker with probability w for sure and some more people. In other words, the probability of hiring a worker jumps upward but the payoff per worker declines by a small amount. This means that this will be a profitable deviation. Now, suppose that $F(w)$ has a hole in the interval $[w_1, w_2]$ with $F(w_1) = F(w_2) < 1$, then an offer of $w_2 - \varepsilon$ attracts the non-captive worker with the same probability but the profits per worker go up and thus it is a profitable deviation.

So we have shown that the distribution of wages, $F(w)$, is well-behaved and has full support over an interval $[w_1, w_2]$. Suppose that $w_1 > b$, then a deviation to b increases profits since at w_1 the probability of trading with the non-captive worker is 0 and thus lowering the wage only increases the profits from the captive worker. This implies that the profits associated with $w = b$ is given by

$$(1-\pi)(z-b).$$

Therefore, as in any mixed strategy equilibrium, any wage offered in equilibrium must satisfy

$$(1-\pi+\pi F(w))(z-w)=(1-\pi)(z-b)$$

This pins down the distribution of wages:

$$F(w)=\frac{(1-\pi)(w-b)}{\pi(z-w)}$$

The upperbound of the distribution is given by

$$1=\frac{(1-\pi)(w_2-b)}{\pi(z-w_2)} \rightarrow w_2=\pi z+(1-\pi)b$$

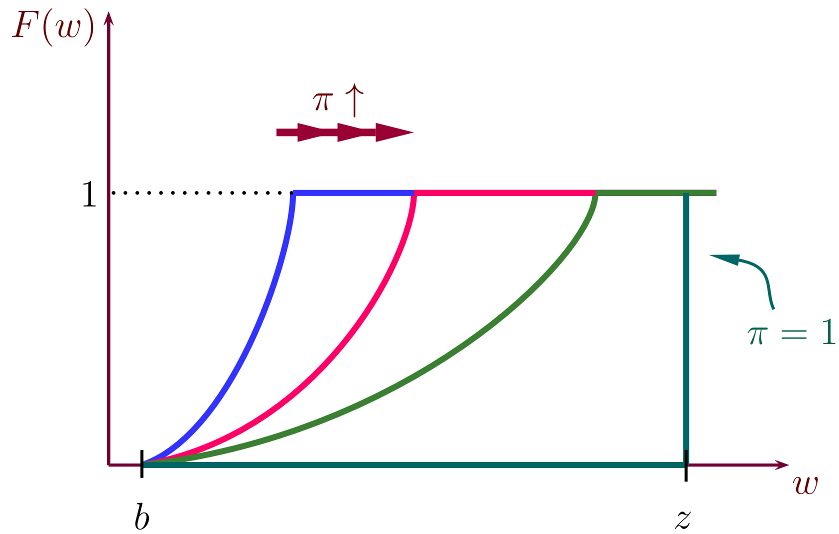

Figure 1: Changes in the distribution of wages in response to change in π

As π converges to 1, the upper bound converges to z and the distribution converges to 0 everywhere except at $w = z$. In other words, the distribution converges to a Dirac delta at $w = z$. The following picture depicts what happens to $F(w)$ as π increases.

Another approach that is very commonly now used in macro is to use Nash bargaining for determination of wages as we show next.

1.3 The Diamond-Mortenson-Pissarides Model

The DMP model, by now is established as the go to model for the analysis of unemployment in macroeconomics. There are many advanced version of it that are being used right now which we won't cover but here I will describe the most basic version.

Consider an economy in continuous time populated by a unit continuum of workers and many firms. The fraction of workers that are unemployed is given by u_t while $1-u_t$ is the fraction of workers that are employed. Firms are assumed to create vacancies whose measure is given by v_t. Given the measure of vacancies and the measure of the unemployed, the number of matches created is $m(u_t, v_t)$. This function is assumed to be constant returns to scale and for now, we assume it is sort of a black box - we do not describe a process through which vacancies match with unemployed workers - if you are interested in seeing some more detailed model, the paper by Petrongolo and Pissarides (2001) is a good resource. A typical assumption is that $m(\cdot,\cdot)$ is Cobb-Douglas: $m(u,v) = Au^{\alpha}v^{1-\alpha}$. It is useful to define job finding rate and vacancy filling rate:

$$\frac{m(u_t,v_t)}{u_t} = m\left(1,\frac{v_t}{u_t}\right) = m(1,\theta_t), \frac{m(u_t,v_t)}{v_t} = \frac{u_t}{v_t}m\left(1,\frac{v_t}{u_t}\right) = \frac{1}{\theta_t}m(1,\theta_t) = q(\theta_t)$$

where $\theta_t = v_t/u_t$ is the market tightness. We assume that firms can open vacancies at a fixed cost of κ and that there is free entry of firms in creating vacancies. Additionally, job separation occurs at rate λ and workers and firms both discount future payoffs at rate ρ.

We will focus on a steady state outcome in which unemployment, market tightness, and wages are constant. Suppose that the equilibrium wage in this steady state is given by w. Let's assume that the value functions associated with an employed worker, unemployed worker, a vacancy and a filled vacancy are given, respectively, by $V(w)$, U, $J(w)$, W. Using the continuous time Bellman equations that we describe above, we have

$$\begin{aligned}
\rho U &= b + \theta q(\theta)[V(w) - U] \\
\rho V(w) &= w + \lambda[U - V(w)] \\
\rho W &= -\kappa + q(\theta)[J(w) - W] \\
\rho J(w) &= z - w + \lambda[W - J(w)]
\end{aligned}$$

Note that free entry implies that $W = 0$. Furthermore, the above imply that

$$J'(w) = V'(w) = \frac{1}{\lambda + \rho}$$

Now, the question is how are wages determined. We use Nash bargaining which is particular type of surplus splitting as determinant of wage. In particular, wages must solve the following optimization

$$\max_w (V(w) - U)^{\gamma}(J(w) - W)^{1-\gamma}$$

Intuitively, $V(w) - U$ is the surplus that the worker gets from the bargaining process and $J(w) - W$ is the surplus allocated to the firm. Under Nash bargaining, wages are determined to solve the above where θ can be thought of as the bargaining power of the workers. This is not very well micro-founded here. In fact, Nash (1950) assumed some axioms that any solution of the bargaining problem must satisfy and showed that it must satisfy the above optimization. Later Rubinstein (1982) showed how to get something similar out of a more explicit game where the two parties make alternating offers and they discount the future.

If we write the FOC associated with the above optimization, we have

$$\frac{\gamma V'(w)}{V(w)-U}+\frac{(1-\gamma)J'(w)}{J(w)-W}=0\Rightarrow\gamma(J(w)-W)=(1-\gamma)(V(w)-U)$$

Finally, note that the law of motion for unemployment is given by

$$\dot{u}_t=(1-u_t)\lambda-u_t\theta_t q(\theta_t)$$

where $(1-u_t)\lambda$ is the inflow of destroyed jobs into unemployment while $u_t\theta_t q(\theta_t)$ is the outflow of newly found jobs out of unemployment. In steady state, we must have

$$(1-u)\lambda-u\theta q(\theta)=0\Rightarrow u=\frac{\lambda}{\lambda+\theta q(\theta)}$$

As v increases, so does θ and the job finding rate $\theta q(\theta)$. This means that the above equation implies a downward relationship between v and u which is referred to as the Beveridge curve.

Now that we have fully described the model, we can solve for some of these objects. We have

$$V(w)=\frac{w+\lambda U}{\rho+\lambda}$$
$$J(w)=\frac{z-w}{\rho+\lambda}$$

Replacing in the bargaining equation:

$$\gamma(z-w)=(1-\gamma)(w-\rho U)\Rightarrow w=\gamma z+(1-\gamma)\rho U$$

The free-entry condition implies that

$$\kappa=q(\theta)J(w)=q(\theta)\frac{z-w}{\rho+\lambda}=q(\theta)(1-\gamma)\frac{z-\rho U}{\rho+\lambda}$$

Thus, the system of equations that need to be solved is given by

$$\kappa(\rho+\lambda)=q(\theta)(1-\gamma)(z-\rho U)$$
$$(\rho+\theta q(\theta))U=b+\theta q(\theta)\frac{\gamma z+(1-\gamma)\rho U+\lambda U}{\rho+\lambda}$$
$$\rho U=b+\theta q(\theta)\frac{\gamma(z-\rho U)}{\rho+\lambda}$$

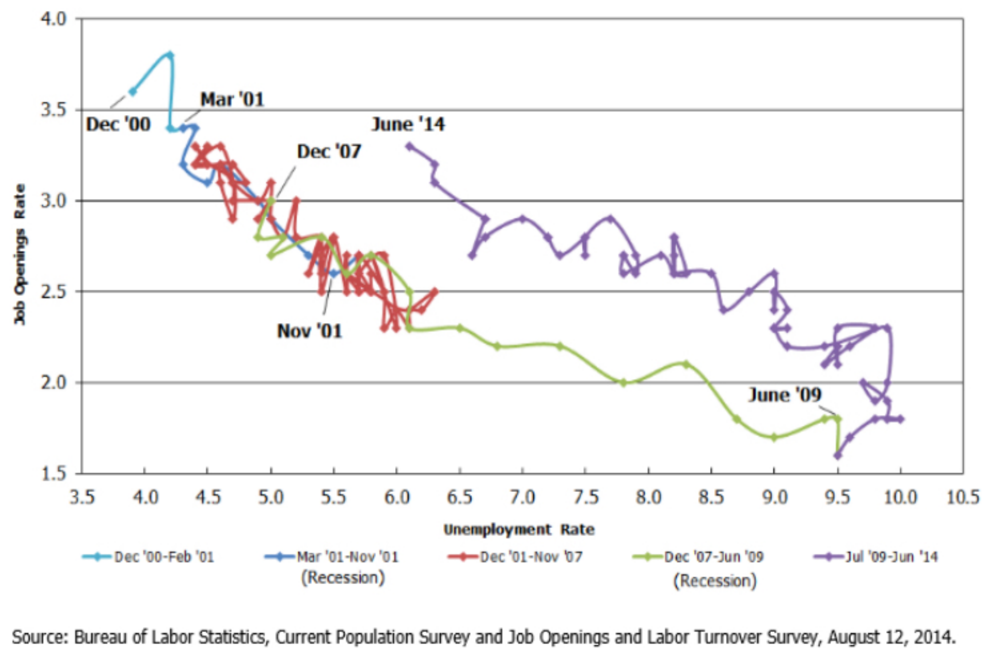

Figure 2: Beveridge Curve Dec 2000 - Jun 2014; from Diamond and Şahin (2015)

In the above formulation $z - \rho U$ is the flow value of total surplus from a match:

$$S = (\rho + \lambda)(J(w) + V(w) - U) = z - w + w + \lambda U - (\rho + \lambda) U$$
$$= z - \rho U$$

The above equations thus can be written as

$$(\rho + \lambda) S = (\rho + \lambda)(z - b) - \theta q(\theta) \gamma S \tag{2}$$
$$\kappa (\rho + \lambda) = q(\theta)(1 - \gamma) S \tag{3}$$

The above are again intuitive. The second is simply comparing the flow value of creating a job to its cost - note that the firm gets a fraction $1 - \gamma$ of the surplus. In addition, the second equation states that flow value of total surplus is equal to surplus from a job minus job finding rate times the workers surplus - this is because when surplus goes, the value of unemployment goes up and the worker likes to wait. The above is a system of two equations and two unknowns. One implies a upward sloping relationship between S and θ - the free entry condition. The other one is a downward sloping relationship between S and θ. It thus have a unique solution. We can also write

$$\frac{\rho + \lambda + \gamma \theta q(\theta)}{(1 - \gamma) q(\theta)} = \frac{z - b}{\kappa}$$

Using this we can perform various comparative statics.

Questions: What is the effect of an increase in unemployment benefits on vacancies? What is the effect of a productivity shock on unemployment and wages? What are the effects of an upward shift in the matching function?

Efficiency of the DMP model

The DMP model is not a competitive equilibrium so it is not guaranteed that the equilibrium is efficient. Obviously if a planner could allocate all the unemployed to vacancies and force firms to create vacancies, then there will be unemployment - as long as $\frac{z-b}{\rho+\lambda} \geq \kappa$. But, let's assume that the planner has to respect the matching technology. Then the problem of maximizing social surplus is given by

$$\max_{u_t, v_t} \int_0^\infty e^{-\rho t} \left[z(1 - u_t) + b u_t - \kappa v_t \right] dt$$

subject to

$$\dot{u}_t = \lambda (1 - u_t) - m(u_t, v_t)$$
$$u_0 : \text{ given}$$

We can use a method similar to the continuous time growth model above to solve this problem. Lagrangian is given by

$$\begin{aligned}
\mathcal{L} &= \int_0^\infty e^{-\rho t}\left\{\left[z\left(1-u_t\right)+bu_t-\kappa v_t\right]-\left[\dot{u}_t-\lambda\left(1-u_t\right)+m\left(u_t,v_t\right)\right]\mu_t\right\}dt \\
&= \int_0^\infty \left\{e^{-\rho t}\left[z\left(1-u_t\right)+bu_t-\kappa v_t\right]-\left[-\lambda\left(1-u_t\right)+m\left(u_t,v_t\right)\right]\mu_t+u_t\left(\dot{\mu}_t-\rho\mu_t\right)\right\}dt \\
&- \mu_0 u_0+\lim_{T\to\infty}\mu_T u_T
\end{aligned}$$

FOCs are given by

$$\begin{aligned}
v_t &: \kappa = m_v\left(u_t,v_t\right)\mu_t \\
u_t &: z-b = \left(\lambda+m_u\left(u_t,v_t\right)\right)\mu_t-\dot{\mu}_t+\rho\mu_t
\end{aligned}$$

In the steady state

$$\kappa = m_v\left(u,v\right)\mu \tag{4}$$

$$z-b = \left(\lambda+\rho+m_u\left(u,v\right)\right)\mu \tag{5}$$

So in the above μ is kind of like the match surplus. It is the shadow cost of unemployment - which is basically the value of a match lost. Couple of notes:

1. When the planner creates a vacancy, the effect of the increase in vacancies on the vacancy filling rate is internalized. This is captured by the term m_v in (4). Compare this with the free entry condition (2). This is a potential source of externality as in the DMP model, when firms create a vacancy they do not internalize the impact this has on other firms - a negative externality. To see this note that

$$\begin{aligned}
q\left(\theta\right) &= m\left(\frac{1}{\theta},1\right) \\
m_v\left(u,v\right) &= \frac{\partial}{\partial v}vm\left(\frac{u}{v},1\right)=\frac{\partial}{\partial v}vq\left(\frac{v}{u}\right) \\
&= q\left(\frac{v}{u}\right)+\frac{v}{u}q'\left(\frac{v}{u}\right)=q\left(\theta\right)+\theta q'\left(\theta\right)
\end{aligned}$$

In other words, what matters for creating vacancy is not only the vacancy filling rate, $q\left(\theta\right)$, but also its derivative $q'\left(\theta\right)$ with respect to market tightness.

2. Similarly, what matters on relating total surplus $z-b$ to match surplus, μ, is not the job finding rate $\theta q\left(\theta\right)$ but the partial derivative of the matching function with respect to u:

$$m_u\left(u,v\right)=\frac{\partial}{\partial u}vq\left(\frac{v}{u}\right)=-\frac{v^2}{u^2}q'\left(\frac{v}{u}\right)=-\theta^2 q'\left(\theta\right)$$

This is a positive externality that vacancy creation has on workers. In other words, firms do not internalize the effect that they have on workers job finding rate.

Now if we divide (5) by (4), we have

$$\frac{z-b}{\kappa}=\frac{\rho+\lambda-\theta^{2}q'\left(\theta\right)}{q\left(\theta\right)+\theta q'\left(\theta\right)}=\frac{1}{q\left(\theta\right)}\frac{\rho+\lambda-\theta^{2}q'\left(\theta\right)}{1+\frac{\theta q'\left(\theta\right)}{q\left(\theta\right)}}$$

Compare this with a similar equation for the DMP equilibrium

$$\frac{z-b}{\kappa}=\frac{\rho+\lambda+\gamma\theta q\left(\theta\right)}{\left(1-\gamma\right)q\left(\theta\right)}$$

In order for the DMP equilibrium to be efficient, the RHS's of the above equations have to be equal. Equating these, we have

$$\frac{\rho+\lambda+\gamma\theta q\left(\theta\right)}{\left(1-\gamma\right)}=\frac{\rho+\lambda-\theta^{2}q'\left(\theta\right)}{1+\frac{\theta q'\left(\theta\right)}{q\left(\theta\right)}}$$

Let $\zeta\left(\theta\right)=-\frac{\theta q'\left(\theta\right)}{q\left(\theta\right)}$. Then we can write the above as

$$\frac{\rho+\lambda+\gamma\theta q\left(\theta\right)}{1-\gamma}=\frac{\rho+\lambda+\zeta\theta q\left(\theta\right)}{1-\zeta}$$

The above implies that efficiency requires

$$\gamma=\zeta$$

This is the so-called Hosios condition as it was first shown by Hosios (1990). In other words, only if the elasticity of the matching function is the same as that of the bargaining power of the worker, then the DMP equilibrium is efficient. It is easy to check that when $\gamma>\zeta$, DMP unemployment is too high and when $\zeta>\gamma$, the DMP unemployment is too low.

One can use the Hosios condition to justify policies that distort the labor market such as minimum wage. See Flinn (2006) for an examination of this.

1.4 Wage Dispersion with Firm and Worker Heterogeneity

So far we have stayed away from heterogeneity of workers and firms in terms of their productivity – while work opportunities have been heterogeneous and random. Here, we discuss the model developed by Postel-Vinay and Robin (2002)(PVR henceforth). This model allows us to examine the dispersion of wages in the data wherein there is significant heterogeneity in wages not fully explained by worker characteristics and partly related to employer characteristics – see Abowd et al. (1999) for an empirical examination of these ideas.

The PVR model relies on the job search and a particular form of wage setting to generate dispersion among seemingly similar workers. The model is consisted of a mass M of workers and a unit continuum of firms.

Workers. Workers are heterogeneous and their type, i.e., labor productivity is given by ε which we assume that has a distribution $H(\varepsilon)$ with density $h(\varepsilon)$ – assume that a density exists everywhere. Moreover, $Supp(H) = [\varepsilon_{min}, \varepsilon_{max}]$. Additionally, workers' discount factors are given by ρ and they have period utility function given by $U(x)$. Moreover, the flow value of unemployment is given by $b\varepsilon$ for some $b < p_{min}$.

Firms. Firms are heterogeneous and their type is given by p with cdf given by $\Gamma(p)$ and pdf given by $\gamma(p)$. If a worker of type ε works for a firm of type p, the output is εp. $Supp(\Gamma) = [p_{min}, p_{max}]$. Note that firms technologies are constant returns to scale w.r.t. to workers so they can produce with as many number of workers of any types. This implies that efficient allocation involves all workers working for the most productive firm.

Matching Technology. Time is continuous and unemployed workers find jobs at rate λ_0. Furthermore, workers find jobs at rate λ_1 while they separate from their jobs at rate δ. Additionally, a worker (employed or unemployed) that finds a new match, draws a firm with productivity p distributed according to cdf $F(p)$ – with pdf $f(p)$.

Wage Setting. Everything is observable to all parties invoved: firms and workers. When a worker that works at a firm with productivity p matches with a firm p', the two firms compete over the worker by playing a limit-pricing game wherein the more productive firm is able to outbid the less productive one and the wages are determined to ensure that the profits of the less productive firm is 0. We formalize this below.

Formally, we can define two value functions for workers: $V_0(\varepsilon)$ for an unemployed worker of type ε and $V(\varepsilon, w, p)$ for a worker of type ε that works for p. Obviously V is increasing in all of its elements.

We also define the following functions: $\phi_0(\varepsilon, p)$, $\phi(\varepsilon, p, p')$

$$V_0(\varepsilon) = V(\varepsilon, \phi_0(\varepsilon, p), p)$$
$$V\left(\varepsilon, \phi\left(\varepsilon, p, p'\right), p'\right) = V(\varepsilon, \varepsilon p, p)$$

In words, ϕ_0 is the wage that makes the worker indifferent between unemployment and having a job at firm p. Since firms have all the bargaining power, i.e., they make take-it-or-leave-it offer, this is the wage that will be offered by firm p to an unemployed worker. $\phi(\varepsilon, p, p')$ is the lowest wage that firm $p' > p$ can offer that makes the worker want to switch from firm p. The RHS of the above equality is the value for a worker of type ε at a firm of type p that pays the worker the maximum possible wage εp, i.e., all the output produced at p. That is if firm p' offers any wage $w' > \phi(\varepsilon, p, p')$, firm p is not able to keep the worker. Under the limit pricing assumption, in this situation the worker switches from p to p' and the wage at p' is given by $\phi(\varepsilon, p, p')$. Therefore, for a worker that currently works at firm p for wage w and matches with a firm p', there are three possibilities:

$p' > p$: In this case, the worker quits her current job and switches to p'. The wage upon switching will be $\phi(\varepsilon, p, p')$

$q(\varepsilon, w, p) \leq p' \leq p$ where $q(\cdot, \cdot, \cdot)$ is defined by

$$\phi(\varepsilon, q(\varepsilon, w, p), p) = w$$

or

$$V(\varepsilon, w, p) = V(\varepsilon, \varepsilon q(\varepsilon, w, p), q(\varepsilon, w, p))$$

In other words, $q(\varepsilon, w, p)$ is the lowest productivity that can offer a wage at least equal to w and break-even. Note that a firm with productivity lower than $q(\varepsilon, w, p)$ is not productive enough to be able to pay w. In this case when $p' \in [q, p]$, the worker stays at the firm yet she will get a raise and her wage increases to $\phi(\varepsilon, p', p)$.

$p' < q(\varepsilon, w, p)$: In this case, the wage stays unchanged and the worker does not move.

We can use the above discussion to fully characterize all value functions and wages. To see this, note that we have

$$\begin{aligned}\rho V_0(\varepsilon) &= U(b\varepsilon) + \lambda_0 \int [V(\varepsilon, \phi_0(\varepsilon, p), p) - V_0(\varepsilon)]\, dF(p) \\ &= U(b\varepsilon) \rightarrow V_0(\varepsilon) = \frac{U(b\varepsilon)}{\rho}\end{aligned}$$

This is because the firms set wages so that an unemployed worker is just indifferent between employment and unemployment. As a result there is no option value of searching for the unemployed.

Vale function for the workers

$$\begin{aligned}\rho V(\varepsilon, w, p) =& U(w) + \lambda_1 \int_{q(\varepsilon,w,p)}^{p} \left[V\left(\varepsilon, \varepsilon p', p'\right) - V(\varepsilon, w, p)\right] dF\left(p'\right) + \\ & \lambda_1 \int_{p}^{p_{max}} [V(\varepsilon, \varepsilon p, p) - V(\varepsilon, w, p)]\, dF\left(p'\right) + \\ & \lambda_0 [V_0(\varepsilon) - V(\varepsilon, w, p)]\end{aligned}$$

Note that working for a firm of type p has an option value. It comes with an option of future wage increases. Now we use the above results about wage setting to solve for the value functions and wages:

Set $w = \varepsilon p$. Intuitively, a worker at p with this wage is at the top of the wage ladder in this firm so any offer from someone else with $p' < p$ will be rejected by the worker. Morevoer, any firm with $p' > p$ will make an offer to the worker so that her valuation remains unchanged. So we get the following:

$$(\rho + \delta) V(\varepsilon, \varepsilon p, p) = U(\varepsilon p) + \delta V_0(\varepsilon) \to V(\varepsilon, \varepsilon p, p) = \frac{U(\varepsilon p) + \delta V_0(\varepsilon)}{\rho + \delta}$$

We then can write

$$\begin{aligned}
(\rho + \delta) V(\varepsilon, w, p) =& U(w) + \delta V_0(\varepsilon) \\
& - \lambda_1 \int_q^p \left[V\left(\varepsilon, \varepsilon p', p'\right) - V(\varepsilon, w, p) \right] d\left(1 - F\left(p'\right)\right) \\
& + \lambda_1 \left[V(\varepsilon, \varepsilon p, p) - V(\varepsilon, w, p) \right] (1 - F(p)) \\
=& U(w) + \delta V_0(\varepsilon) \\
& - \lambda_1 \left[V\left(\varepsilon, \varepsilon p', p'\right) - V(\varepsilon, w, p) \right] \left(1 - F\left(p'\right)\right) \Big|_q^p \\
& + \lambda_1 \int_q^p \left(1 - F\left(p'\right)\right) \frac{\varepsilon U'\left(\varepsilon p'\right)}{\rho + \delta} dp' \\
& + \lambda_1 \left[V(\varepsilon, \varepsilon p, p) - V(\varepsilon, w, p) \right] (1 - F(p)) \\
=& U(w) + \delta V_0(\varepsilon) \\
& + \lambda_1 \int_q^p \left(1 - F\left(p'\right)\right) \frac{\varepsilon U'\left(\varepsilon p'\right)}{\rho + \delta} dp'
\end{aligned}$$

Now we can use the definition of ϕ adn write

$$\begin{aligned}
(\rho + \delta) V(\varepsilon, \varepsilon \hat{p}, \hat{p}) &= U(\phi(\varepsilon, \hat{p}, p)) + \delta V_0(\varepsilon) + \lambda_1 \int_{\hat{p}}^p (1 - F(x)) \frac{\varepsilon U'(\varepsilon x)}{\rho + \delta} dx \\
U(\varepsilon \hat{p}) &= U(\phi(\varepsilon, \hat{p}, p)) + \lambda_1 \int_{\hat{p}}^p (1 - F(x)) \frac{\varepsilon U'(\varepsilon x)}{\rho + \delta} dx \\
U(\phi(\varepsilon, \hat{p}, p)) &= U(\varepsilon \hat{p}) - \lambda_1 \int_{\hat{p}}^p (1 - F(x)) \frac{\varepsilon U'(\varepsilon x)}{\rho + \delta} dx
\end{aligned}$$

This is the key equation of the paper that determines someone wage. In other words, we can say

$$w_t = \phi(\varepsilon, \hat{p}_t, p_t)$$

where p_t is the current productivity of a worker's employer and $\hat{p}_t$ is the last most productive firm to make her an offer. Moreover, when $U(x) = \frac{x^{1-\alpha}-1}{1-\alpha}$ or $U(x) = \log x$, we can write the above as

$$\log \phi(\varepsilon, \hat{p}, p) = \log \varepsilon + \frac{1}{1-\alpha} \log \left[\hat{p} - \frac{\lambda_1}{\rho + \delta} \int_{\hat{p}}^{p} (1 - F(x)) x^{-\alpha} dx \right], \quad \alpha \neq 1$$

$$\log \phi(\varepsilon, \hat{p}, p) = \log \varepsilon + \log \hat{p} - \frac{\lambda_1}{\rho + \delta} \int_{\hat{p}}^{p} \frac{1 - F(x)}{x} dx$$

One thing to notice is that the wage is below $\varepsilon \hat{p}$ exactly because of the option value. We also see that wage profiles are concave and this is because as workers move up the job ladder, the option values decline. Quite intuitively, the higher is the rate of getting an offer on the job λ_1 the higher is this option value and the lower is the wage.

Steady State: In order to take the model to the data, we need to impose some discipline on the aggregates and the easiest way of doing this would be to assume stationarity. Then, we have

unemployment rate

$$(1-u)\delta - \lambda_0 u = 0 \rightarrow u = \frac{\delta}{\delta + \lambda_0}$$

Number of workers at firms p: cdf: $L(p)$

$$-L(p)[\delta + \lambda_1(1 - F(p))](1-u) + uF(p)\lambda_0 = 0 \rightarrow L(p) = \frac{\delta F(p)}{\delta + \lambda_1(1 - F(p))}$$

Number of workers of type ε at p: pdf: $\ell(\varepsilon, p)$ given by

$$\ell(\varepsilon, p) = h(\varepsilon)\ell(p) = h(\varepsilon) L'(p) = \frac{\delta(\delta + \lambda_1) f(p)}{(\delta + \lambda_1(1 - F(p)))^2}$$

Distribution of wages conditional on (ε, p): cdf: $G(w|\varepsilon, p)$

$$-G(w|\varepsilon, p)\ell(p) h(\varepsilon)(1-u)(\delta + \lambda_1(1 - F(q(\varepsilon, w, p))))$$

$$+\lambda_0 u h(\varepsilon) f(p) + \lambda_1 f(p)(1-u) h(\varepsilon) \int_{p_{min}}^{q(\varepsilon, w, p)} \ell(p') dp' = 0$$

$$-G(w|\varepsilon, p) \frac{\delta(\delta + \lambda_1)(\delta + \lambda_1(1 - F(q)))}{(\delta + \lambda_1(1 - F(p)))^2}$$

$$+\delta + \lambda_1 \frac{\delta F(q)}{\delta + \lambda_1(1 - F(q))} = 0$$

$$\frac{(\delta + \lambda_1(1 - F(p)))^2}{(\delta + \lambda_1(1 - F(q)))^2} = G(w|\varepsilon, p)$$

Note that since $b < p_{min}$, there is a flat in $G(w|\varepsilon, p)$. We can also write the above as

$$G(\phi(\varepsilon, \hat{p}, p)|\varepsilon, p) = \frac{(\delta + \lambda_1(1 - F(p)))^2}{(\delta + \lambda_1(1 - F(\hat{p})))^2}$$

I am going to leave the econometrics out but the authors structurally estimate this model – which is quite difficult given the nature of the data they have. The main result is that person effects explain only a small fraction of the variation in wages.

References

Abowd, J. M., F. Kramarz, and D. N. Margolis (1999): "High wage workers and high wage firms," *Econometrica*, 67, 251–333.

Burdett, K. and K. Judd (1983): "Equilibrium Price Dispersion," *Econometrica*, 51, 955–69.

Diamond, P. A. (1971): "A Model of Price Adjustment," *Journal of Economic Theory*, 3, 156–168.

Diamond, P. A. and A. Şahin (2015): "Shifts in the Beveridge curve," *Research in Economics*, 69, 18–25.

Flinn, C. J. (2006): "Minimum wage effects on labor market outcomes under search, matching, and endogenous contact rates," *Econometrica*, 74, 1013–1062.

Hornstein, A., P. Krusell, and G. L. Violante (2011): "Frictional wage dispersion in search models: A quantitative assessment," *The American Economic Review*, 101, 2873–2898.

Hosios, A. J. (1990): "On the efficiency of matching and related models of search and unemployment," *The Review of Economic Studies*, 57, 279–298.

Nash, J. F. (1950): "The Bargaining Problem," *Econometrica*, 18, 155–162.

Petrongolo, B. and C. A. Pissarides (2001): "Looking into the black box: A survey of the matching function," *Journal of Economic literature*, 39, 390–431.

Postel-Vinay, F. and J.-M. Robin (2002): "Equilibrium wage dispersion with worker and employer heterogeneity," *Econometrica*, 70, 2295–2350.

Rubinstein, A. (1982): "Perfect Equilibrium in a Bargaining Model," *Econometrica*, 50, 97–109.