47802 - Macroeconomics I

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Midterm Exam's Solution

You have 110 minutes

Solve one of problem 1 and 2 as well as problem 3.

1. An Economy with Heterogeneity

Consider an infinite-horizon economy with a single consumption good, c, in every period and leisure, ℓ.

The economy is comprised of 2 consumers. Each has preference of the following form:

$$\sum_{t=0}^{\infty} \beta^t [u(c_{t,i}) + \gamma_i v(\ell_{t,i})], \quad i = 1, 2$$

where u and v are continuously differentiable, strictly increasing and strictly concave functions and $\gamma_1 > \gamma_2 > 0$. Suppose also that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and $v(\ell) = \psi \frac{\ell^{1-\sigma}}{1-\sigma}$ for $\sigma \neq 1$. When $\sigma = 1$, $u(c) = \log c$ and $v(\ell) = \psi \log \ell$.

Each agent in every period is endowed with 1 unit of leisure and 0 units of consumption good. There is a single firm with the following production technology:

$$Y = \left\{(n, y); (n, y) \in \mathbb{R}_+^2, y \leq An\right\}$$

for some $A > 0$.

a. (5 points) Define an Arrow-Debreu equilibrium for this economy.

> **Solution.** An AD eequilibrium for this economy is consisted of allocations $\{c_{t,i}, \ell_{t,i}\}_{t=0,1,\cdots,i=1,2}$ and sequence of prices and wages $\{p_t, w_t\}_{t=0,1,\cdots}$ such that:
>
> - Given prices and wages, consumers 1 and 2 solve the following problem:
>
> $$\max_{\{c_{t,i},\ell_{t,i}\}_{t=0,\cdots}} \sum_{t=0}^{\infty} \beta^t \left[u(c_{t,i}) + \gamma_i v(\ell_{t,i})\right]$$
>
> subject to
>
> $$\sum_{t=0}^{\infty} p_t c_{t,i} \leq \sum_{t=0}^{\infty} w_t (1 - \ell_{t,i})$$
> $$0 \leq c_{t,i},$$
> $$0 \leq \ell_{t,i} \leq 1$$
>
> - Given prices and wages, the firm maximizes
>
> $$\max_n p_t An - w_t n$$

- Allocations are feasible:

$$c_{t,1} + c_{t,2} = A\left(2 - \ell_{t,1} - \ell_{t,2}\right)$$

b. (5 points) Assume an interior equilibrium. Calculate the prices explicitly.

Solution. Each consumer's optimization problem is a strictly concave problem and therefore has a unique solution which must satisfy the following first order conditions - since allocations are interior:

$$\beta^t u'\left(c_{t,i}\right) = \lambda_i p_t \Rightarrow \beta^t \left(c_{t,i}\right)^{-\sigma} = \lambda_i p_t$$
$$\beta^t \gamma_i v'\left(\ell_{t,i}\right) = \lambda_i w_t \Rightarrow \beta^t \psi \gamma_i \left(\ell_{t,i}\right)^{-\sigma} = \lambda_i w_t$$

From the firm's FOC, we have

$$p_t A = w_t$$

Therefore, we have

$$c_{t,i} = \left(\frac{\beta^t}{\lambda_i p_t}\right)^{\frac{1}{\sigma}}$$
$$\ell_{t,i} = \left(\frac{\beta^t \psi \gamma_i}{\lambda_i A p_t}\right)^{\frac{1}{\sigma}}$$

Replacing the above in the budget constraints, we will have

$$\sum_{t=0}^{\infty} p_t c_{t,i} = \lambda_i^{-\frac{1}{\sigma}} \sum_{t=0}^{\infty} p_t^{1-\frac{1}{\sigma}} \beta^{\frac{t}{\sigma}}$$
$$\sum_{t=0}^{\infty} p_t \left(1 - \ell_{t,i}\right) = \sum_{t=0}^{\infty} p_t - \left(\frac{\psi \gamma_i}{\lambda_i A}\right)^{\frac{1}{\sigma}} \sum_{t=0}^{\infty} p_t^{1-\frac{1}{\sigma}} \beta^{\frac{t}{\sigma}}$$

Equating the above two, we have

$$\lambda_i^{-\frac{1}{\sigma}} \left[1 + \left(\frac{\psi \gamma_i}{A}\right)^{\frac{1}{\sigma}}\right] \sum_{t=0}^{\infty} p_t^{1-\frac{1}{\sigma}} \beta^{\frac{t}{\sigma}} = \sum_{t=0}^{\infty} p_t$$
$$\lambda_i^{-\frac{1}{\sigma}} = \frac{\sum_{t=0}^{\infty} p_t}{\sum_{t=0}^{\infty} p_t^{1-\frac{1}{\sigma}} \beta^{\frac{t}{\sigma}}} \frac{1}{1 + \left(\frac{\psi \gamma_i}{A}\right)^{\frac{1}{\sigma}}}$$

and as a result,

$$c_{t,i} = \left(\frac{\beta^t}{p_t}\right)^{\frac{1}{\sigma}} \frac{\sum_{t=0}^{\infty} p_t}{\sum_{t=0}^{\infty} p_t^{1-\frac{1}{\sigma}} \beta^{\frac{t}{\sigma}}} \frac{1}{1 + \left(\frac{\psi \gamma_i}{A}\right)^{\frac{1}{\sigma}}} \tag{1}$$
$$\ell_{t,i} = \left(\frac{\beta^t}{p_t}\right)^{\frac{1}{\sigma}} \frac{\sum_{t=0}^{\infty} p_t}{\sum_{t=0}^{\infty} p_t^{1-\frac{1}{\sigma}} \beta^{\frac{t}{\sigma}}} \frac{\left(\frac{\psi \gamma_i}{A}\right)^{\frac{1}{\sigma}}}{1 + \left(\frac{\psi \gamma_i}{A}\right)^{\frac{1}{\sigma}}} \tag{2}$$

Replacing in the market clearing, we have

$$\left(\frac{\beta^t}{p_t}\right)^{\frac{1}{\sigma}} \frac{\sum_{t=0}^{\infty} p_t}{\sum_{t=0}^{\infty} p_t^{1-\frac{1}{\sigma}} \beta^{\frac{t}{\sigma}}} \left(\frac{1}{1+\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}} + \frac{1}{1+\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}} + \frac{A\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}} + \frac{A\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}} \right) = 2A$$

By Walras' law, we can normalize any value related to prices to 1. We can thus assume that

$$\frac{\sum_{t=0}^{\infty} p_t}{\sum_{t=0}^{\infty} p_t^{1-\frac{1}{\sigma}} \beta^{\frac{t}{\sigma}}} = 1$$

which implies that

$$p_t = \beta^t (2A)^{-\sigma} \left(\frac{1}{1+\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}} + \frac{1}{1+\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}} + \frac{A\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}} + \frac{A\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}} \right)^{\sigma}$$

$$w_t = A\beta^t (2A)^{-\sigma} \left(\frac{1}{1+\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}} + \frac{1}{1+\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}} + \frac{A\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}} + \frac{A\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}} \right)^{\sigma}$$

c. (10 points) Show that $c_{t,1} < c_{t,2}$ and $\ell_{t,1} > \ell_{t,2}$.

Solution. If we use equation (1) and divide the one associated with consumer 1 by that of consumer 2, we have

$$\frac{c_{t,2}}{c_{t,1}} = \frac{1+\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}}$$

Since $\sigma > 0$ and $\gamma_1 > \gamma_2$, $\gamma_1^{\frac{1}{\sigma}} > \gamma_2^{\frac{1}{\sigma}}$. This implies that the above ratio is bigger than 1.

Similarly and for leisure, we have

$$\frac{\ell_{t,2}}{\ell_{t,1}} = \frac{\gamma_2^{\frac{1}{\sigma}}}{\gamma_1^{\frac{1}{\sigma}}} \frac{1+\left(\frac{\psi\gamma_1}{A}\right)^{\frac{1}{\sigma}}}{1+\left(\frac{\psi\gamma_2}{A}\right)^{\frac{1}{\sigma}}} = \frac{\gamma_1^{-\frac{1}{\sigma}} + \left(\frac{\psi}{A}\right)^{\frac{1}{\sigma}}}{\gamma_2^{-\frac{1}{\sigma}} + \left(\frac{\psi}{A}\right)^{\frac{1}{\sigma}}}$$

Since $\gamma_1 > \gamma_2$ and $\sigma > 0$, the above ratio is less than 1. This proves the claim.

d. (10 points) Now suppose that $\sigma = 1$. Set up and solve the social planner's problem. Find a condition on welfare weights so that the result in part c still holds.

Solution. The social planner's problem is given by

$$\max \alpha \sum_{t=0}^{\infty} \beta^t \left[\log c_{t,1} + \psi\gamma_1 \log \ell_{t,1}\right] + (1-\alpha) \sum_{t=0}^{\infty} \beta^t \left[\log c_{t,2} + \psi\gamma_2 \log \ell_{t,2}\right]$$

subject to

$$c_{t,1} + c_{t,2} + A\ell_{t,1} + A\ell_{t,2} = 2A$$

The FOC's are given by

$$\frac{\alpha\beta^t}{c_{t,1}} = \lambda_t = \frac{(1-\alpha)\,\beta^t}{c_{t,2}}$$
$$\psi\gamma_1\frac{\alpha\beta^t}{\ell_{t,1}} = A\lambda_t = \psi\gamma_2\frac{(1-\alpha)\,\beta^t}{\ell_{t,2}}$$

where λ_t is the lagrange for feasibility constraint at t. We have

$$c_{t,1} = \frac{\alpha\beta^t}{\lambda_t}, c_{t,2} = \frac{(1-\alpha)\,\beta^t}{\lambda_t}$$
$$\ell_{t,1} = \frac{\psi\gamma_1}{A}\frac{\alpha\beta^t}{\lambda_t}, \ell_{t,2} = \frac{\psi\gamma_2}{A}\frac{(1-\alpha)\,\beta^t}{\lambda_t}$$

Replacing, in feasibility,

$$\frac{\beta^t}{\lambda_t}\left[1+\psi\gamma_1\alpha+\psi\gamma_2\,(1-\alpha)\right] = 2A \Rightarrow \lambda_t = \beta^t\frac{1+\psi\gamma_1\alpha+\psi\gamma_2\,(1-\alpha)}{2A}$$

Now, the ratio of the consumption of 1 to 2, in the above allocation is $\frac{\alpha}{1-\alpha}$. The same ratio in the AD equilibrium is

$$\frac{1+\frac{\psi\gamma_2}{A}}{1+\frac{\psi\gamma_1}{A}}$$

For these values to coincide, we must have

$$\frac{\alpha}{1-\alpha} = \frac{1+\frac{\psi\gamma_2}{A}}{1+\frac{\psi\gamma_1}{A}} \Rightarrow \alpha = \frac{1+\frac{\psi\gamma_2}{A}}{2+\frac{\psi(\gamma_1+\gamma_2)}{A}}$$

2. Dynamic Programming

Consider the following sequence problem:

$$\max_{c_t,k_{t+1}} \sum_{t=0}^{\infty}\beta^t u(c_t)$$

s.t.

$$c_t + k_{t+1} \leq F(k_t)$$
$$k_{t+1} \geq 0 \ \ \forall t \geq 0,$$
$$k_0 : \text{ given}$$

a. (5 points) Write this problem in canonical form.

Solution. Let $\Gamma(k)$ be defined as

$$\Gamma(k) = [0, F(k)]$$

and let

$$G(k, k') = u(F(k) - k')$$

Then, the above problem in canonical form is given by

$$\max_{\{k_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t G(k_t, k_{t+1})$$

subject to

$$k_{t+1} \in \Gamma(k_t), k_0 : \text{ given}$$

b. (15 points) Assume $u(c) = -e^{-\alpha c}$ and $F(k) = Ak$ where $A > 1$. Write the Bellman equation and solve for the value function. Hint: Guess that $v(k) = -Ce^{-\gamma k}$ for some value of C and γ and verify. Ignore the positive consumption constraint.

Solution. The Bellman equation for this problem is given by

$$v(k) = \max_{k'} -e^{-\alpha(Ak-k')} + \beta v(k')$$

If we guess that $v(k) = -Ce^{-\gamma k}$, then the optimization in the RHS of the above is given by

$$\max_{k'} -e^{-\alpha(Ak-k')} - \beta Ce^{-\gamma k'}$$

The solution of this problem must satisfy the following FOC:

$$-\alpha e^{-\alpha(Ak-k')} + \gamma\beta Ce^{-\gamma k'} = 0 \Rightarrow \gamma\beta Ce^{-\gamma k'} = \alpha e^{-\alpha(Ak-k')}$$

Taking logs, we have

$$\begin{aligned} -\gamma k' + \log\gamma + \log\beta + \log C &= \log\alpha - \alpha(Ak - k') \\ \frac{\log\gamma + \log\beta + \log C - \log\alpha}{\gamma + \alpha} + \frac{\alpha}{\alpha+\gamma}Ak &= k' \end{aligned}$$

Evaluated at this value, the objective is given by

$$\begin{aligned} -e^{-\alpha(Ak-k')} - \beta Ce^{-\gamma k'} &= -e^{-\frac{\alpha\gamma}{\alpha+\gamma}Ak - \alpha\frac{\log\gamma+\log\beta+\log C-\log\alpha}{\gamma+\alpha}} - \beta Ce^{-\frac{\alpha\gamma}{\alpha+\gamma}Ak - \gamma\frac{\log\gamma+\log\beta+\log C-\log\alpha}{\gamma+\alpha}} \\ &= -e^{-\frac{\alpha\gamma}{\alpha+\gamma}Ak}\left[e^{-\alpha\frac{\log\gamma+\log\beta+\log C-\log\alpha}{\gamma+\alpha}} + \beta Ce^{-\gamma\frac{\log\gamma+\log\beta+\log C-\log\alpha}{\gamma+\alpha}}\right] \end{aligned}$$

In order for the above to have the same form as $-Ce^{-\gamma k}$, we must have

$$\begin{aligned} \gamma &= \frac{\alpha\gamma}{\alpha+\gamma}A \Rightarrow \alpha + \gamma = \alpha A \Rightarrow \gamma = \alpha(A-1) \\ C &= e^{-\alpha\frac{\log\gamma+\log\beta+\log C-\log\alpha}{\gamma+\alpha}} + \beta Ce^{-\gamma\frac{\log\gamma+\log\beta+\log C-\log\alpha}{\gamma+\alpha}} \\ &= e^{-\frac{\log(A-1)+\log\beta+\log C}{A}} + \beta Ce^{-(A-1)\frac{\log(A-1)+\log\beta+\log C}{A}} \end{aligned}$$

C must solve the bottom equation.

c. (10 points) Assume $u(\cdot)$ and $F(\cdot)$ are strictly concave and continuously differentiable. Assume that the solution to the RHS of Bellman equation is interior. Show that policy function is increasing in capital. You may use theorems from Stockey-Lucas-Prescott as long as you state them and show that it is valid to use them. Moreover, assume that the solution of the Bellman equation exists and is bounded and continuous.

Solution. Using theorems in SLP, we know that the unique solution of the Bellman equation must be strictly concave and differentiable. Since it is assumed that the value of k' that solve the RHS of the Bellman equation, it must satisfy the following FOC

$$u'\left(F\left(k\right)-k'\right)=\beta v'\left(k'\right) \tag{3}$$

Suppose that $k_1 < k_2$ and their associated future value of capital satisfies $k_2' < k_1'$ - contrary to the claim. The strict concavity of the value function implies that

$$\beta v'\left(k_2'\right) > \beta v'\left(k_1'\right)$$

In addition, we have

$$F\left(k_2\right)-k_2' > F\left(k_1\right)-k_1'$$

Strict concavity of the utility function implies that u' is strictly decreasing and therefore

$$u'\left(F\left(k_2\right)-k_2'\right) < u'\left(F\left(k_1\right)-k_1'\right)$$

Since k_2' is an optimal choice under k_2, equation (3) implies that

$$u'\left(F\left(k_2\right)-k_2'\right)=\beta v'\left(k_2'\right)$$

By above inequalities, we have

$$u'\left(F\left(k_1\right)-k_1'\right) > u'\left(F\left(k_2\right)-k_2'\right)=\beta v'\left(k_2'\right) > \beta v'\left(k_1'\right)$$

The above inequality means that the pair (k_1, k_1') cannot satisfy (3) which means k_1' cannot be optimal under k_1.

3. Optimal Taxation with Endogenous Government Spending

Consider the baseline one-sector growth model but assume that in each period there is an additional consumption good: a public good that is provided by the government. Suppose that utility of the representative household is given by

$$u\left(c,g,\ell\right)=\frac{c^{1-\sigma}}{1-\sigma}+\psi\frac{g^{1-\sigma}}{1-\sigma}+\zeta\frac{\ell^{1-\sigma}}{1-\sigma}$$

when $\sigma \neq 1$ and the utility for $\sigma = 1$ is defined as usual - see problem 1.

Assume that the relative price of the public good to private consumption is 1. Production of the public and private goods are done in firms using a Cobb-Douglas production function which is the same for both goods.

Households are standard. They own the capital stock and are endowed with leisure. They provide labor and rent out capital to firms and they consume the two consumption good. Note that households do not purchase the public good - this is determined by the government.

a. (5 points) Define a TDCE for this economy assuming that the government can impose linear taxes on households - their various sources of income and consumption of the consumption good as well as investment in physical capital.

Solution. A TDCE for this problem is given by a sequence of allocations, $\{c_t, \ell_t, k_{t+1}, x_t\}_{t=0}^{\infty}$, prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, and policies $\{\tau_{l,t}, \tau_{x,t}, \tau_{c,t}, \tau_{k,t}\}_{t=0}^{\infty}$ and sequence of government purchases $\{g_t\}_{t=0}^{\infty}$ such that

- Given the sequence of prices and policies, the sequence of allocations is a solution of the following optimization problem

$$\max \sum_{t=0}^{\infty} \beta^t u\left(c_t, g_t, \ell_t\right)$$

subject to

$$\sum_{t=0}^{\infty} p_t\left[\left(1+\tau_{c,t}\right)c_t + \left(1+\tau_{x,t}\right)x_t\right] \leq \sum_{t=0}^{\infty}\left[w_t\left(1-\tau_{l,t}\right)\left(1-\ell_t\right) + r_t\left(1-\tau_{k,t}\right)k_t\right]$$
$$k_{t+1} = k_t\left(1-\delta\right) + x_t$$
$$k_0 : \text{ given}$$

- Firms are as usual
- Government budget constraint is satisfied

$$\sum_{t=0}^{\infty} p_t\left(\tau_{c,t}c_t + \tau_{x,t}x_t\right) + \sum_{t=0}^{\infty}\tau_{l,t}w_t\left(1-\ell_t\right) + \tau_{k,t}r_tk_t = \sum_{t=0}^{\infty} p_tg_t + b_0$$

b. (5 points) Derivate the implementability condition for this economy and show that an allocation is part of a TDCE if and only if it is feasible and satisfies the implementability condition.

Solution. The derivation of IC is standard. Refer to the notes for this derivation. IC is given by

$$\sum_{t=0}^{\infty}\beta^t\left[c_t^{1-\sigma} - \zeta\ell_t^{-\sigma}\left(1-\ell_t\right)\right] = c_0^{-\sigma}k_0\frac{F_{K,0}\left(1-\tau_{k,0}\right) + \left(1-\delta\right)\left(1+\tau_{x,0}\right)}{1+\tau_{c,0}}$$

Note: we have to assume away taxes on investment and consumption since with them, it is optimal to set $\tau_{c,0}$ equal to a large enough number to fully finance all government spending.

c. (5 points) Formulate the Ramsey problem.

Solution. The Ramsey problem is given by

$$\max_{c_t,\ell_t,k_{t+1},x_t,g_t} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \psi \frac{g_t^{1-\sigma}}{1-\sigma} + \zeta \frac{\ell_t^{1-\sigma}}{1-\sigma} \right]$$

subject to

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \left[c_t^{1-\sigma} - \zeta \ell_t^{-\sigma} (1-\ell_t) \right] &= c_0^{-\sigma} k_0 \left[F_{K,0} (1-\tau_{k,0}) + (1-\delta) \right] \\ c_t + g_t + k_{t+1} &= F(k_t, 1-\ell_t) + k_t (1-\delta) \\ \tau_{k,0} &\leq 1 \end{aligned}$$

d. (10 points) Calculate the long-ratio of optimal government spending to GDP.

Solution. The FOC associated with the above is given by

$$\beta^t c_t^{-\sigma} \left[1 - \lambda (1-\sigma) \right] = \mu_t \tag{4}$$

$$\beta^t \zeta \ell_t^{-\sigma} \left[1 - \lambda \left((1-\sigma) + \sigma \ell_t^{-1} \right) \right] = F_n (k_t, 1-\ell_t) \mu_t \tag{5}$$

$$\mu_t \left[F_k (k_t, 1-\ell_t) + 1 - \delta \right] = \mu_{t-1} \tag{6}$$

$$\beta^t \psi g_t^{-\sigma} = \mu_t \tag{7}$$

for all $t \geq 1$. For $t = 0$,

$$\begin{aligned} c_0^{-\sigma} \left[1 - \lambda (1-\sigma) \right] - \sigma \lambda c_0^{-\sigma-1} k_0 \left[F_{k,0} (1-\tau_{k,0}) + 1 - \delta \right] &= \mu_0 \\ \zeta \ell_0^{-\sigma} \left[1 - \lambda \left((1-\sigma) - \sigma \ell_0^{-1} \right) \right] - \lambda c_0^{-\sigma} k_0 F_{kn,0} (1-\tau_{k,0}) &= F_n (k_t, 1-\ell_t) \mu_t \\ -\lambda c_0^{-\sigma} k_0 F_{k,0} (1-\tau_{k,0}) &= 0 \end{aligned}$$

In other words, either $\tau_{k,0} = 1$ or $\lambda = 0$. If $\lambda = 0$, then consumption government spending ratio is given by

$$\frac{g_t}{c_t} = \psi^{\frac{1}{\sigma}} \rightarrow \frac{g_t}{g_t + c_t} = \frac{\psi^{\frac{1}{\sigma}}}{\psi^{\frac{1}{\sigma}} + 1}$$

In the long-run

$$x_t = \delta k_t$$

and capital taxes are zero. Therefore

$$\beta \left[\alpha \frac{y_t}{k_t} + 1 - \delta \right] = 1$$

where we have used Cobb-Douglass assumption. Hence,

$$\frac{y_t}{k_t} = \frac{1}{\alpha} \left(\frac{1}{\beta} - 1 + \delta \right)$$

Hence,

$$\frac{x_t}{y_t} = \delta \frac{k_t}{y_t} = \frac{\delta\alpha}{1/\beta - 1 + \delta}$$

Therefore

$$\frac{c_t + g_t}{y_t} = 1 - \frac{\delta\alpha}{1/\beta - 1 + \delta}$$

and

$$\frac{g_t}{y_t} = \frac{\psi^{\frac{1}{\sigma}}}{\psi^{\frac{1}{\sigma}} + 1} \frac{1/\beta - 1 + \delta\left(1-\alpha\right)}{1/\beta - 1 + \delta}$$

If $\tau_{k,0} = 1$, then

$$\frac{g_t}{c_t} = \frac{\psi^{\frac{1}{\sigma}}}{\psi^{\frac{1}{\sigma}} + \left[1 - \lambda\left(1-\sigma\right)\right]^{\frac{1}{\sigma}}} \tag{8}$$

$$\frac{g_t}{y_t} = \frac{\psi^{\frac{1}{\sigma}}}{\psi^{\frac{1}{\sigma}} + \left[1 - \lambda\left(1-\sigma\right)\right]^{\frac{1}{\sigma}}} \frac{1/\beta - 1 + \delta\left(1-\alpha\right)}{1/\beta - 1 + \delta}$$

where λ is a calculated in order to satisfy the implementability constraint. Whether λ is zero or positive depends on the strength of preferences for the public good, ψ. For low values of ψ, initial taxation of capital is enough to generate enough revenue for the government to finance government spending. On the other hand, when ψ is high, setting $\tau_{k,0}$ to 1 is not enough to finance the desired level of g_t.

e. (10 points) What is the long-run value of optimal capital income taxes? What is the long-run value of optimal labor income taxes?

Solution. Similar to the notes, the long-run value of optimal capital income taxes is 0. As for long-run labor income taxes, there is no explicit solution for it. We have

$$\frac{\tau_l^*}{1-\tau_l^*} = \frac{1}{\ell^*} \frac{\sigma\lambda}{1 - \lambda\left(1-\sigma\right)}$$

Note that

$$\frac{y^*}{k^*} = A\left(\frac{1-\ell}{k}\right)^{1-\alpha} = \frac{1/\beta - 1 + \delta}{\alpha} \Rightarrow \frac{k}{1-\ell} = \left(\frac{\alpha A}{1/\beta - 1 + \delta}\right)^{\frac{1}{1-\alpha}} \tag{9}$$

As a result

$$F_n = \left(1-\alpha\right) A \left(\frac{k}{1-\ell}\right)^{\alpha} = \left(1-\alpha\right) A \left(\frac{\alpha A}{1/\beta - 1 + \delta}\right)^{\frac{\alpha}{1-\alpha}}$$

where ℓ^* and c^* solve the following system of equations:

$$\frac{\left(\ell^*\right)^{-\sigma}}{\left(c^*\right)^{-\sigma}} \frac{1 - \lambda\left(\left(1-\sigma\right) + \sigma/\ell^*\right)}{1 - \lambda\left(1-\sigma\right)} = \frac{A}{\zeta}\left(1-\alpha\right)\left(\frac{\alpha A}{1/\beta - 1 + \delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{10}$$

$$c^*\left(1 + \frac{\psi^{\frac{1}{\sigma}}}{\psi^{\frac{1}{\sigma}} + \left[1 - \lambda\left(1-\sigma\right)\right]^{\frac{1}{\sigma}}}\right) + \delta\left(1-\ell^*\right)\left(\frac{\alpha A}{1/\beta - 1 + \delta}\right)^{\frac{1}{1-\alpha}} = \left(1-\ell^*\right) A \left(\frac{\alpha A}{1/\beta - 1 + \delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{11}$$

where the first equation is the trade-off between consumption and leisure from (4) and (7) while the second is feasibility, where we replaced for g^* from (8) and for k^* from (9). One can solve (11) for c^* as a linear function of $1 - \ell^*$ and replace in (10) to have one equation one unknown in ℓ^*. Note that when $\sigma = 1$, we can solve the above for ℓ^* in closed form.