Contracts in Macroeconomics

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1 Contracts and Mechanism Design

1.1 Introduction

- Economics is about allocation of resources and gain from trade: contracts are essential
- Contracts?
  - Competitive equilibrium; linear contracts
  - households paying taxes; social contract
  - Electoral college system for voting
- Standard approach in macro: Arrow-Debreu competitive markets
  - ignores incentive problems:
    - Poeple know a lot
    - They can commit to terms
- This course: we relax those assumptions to some extent.
- Main focus: private information and limited commitment
- Main tool: mechanism design
- Applications:
  - Optimal Taxation
  - Financial Contracting

1.2 Example

- Consider an economy with two homogenous goods: apples, c_a, and bananas, c_b
- There are two agents:
  - Each can be of two types: $\theta = \theta_H > \theta_L$; $\Pr(\theta_H) = \pi$.
  - θ is taste for apple; Preferences

$$\theta u(c_a) + u(c_b)$$

  - Shocks are i.i.d across households.
  - Joint distribution of shocks: $\mu(\theta_1, \theta_2) = \Pr(\theta_1)\Pr(\theta_2)$
- everyone has the same endowment of apples and oranges, e_a, e_b
- Allocations:

$$c_a^i(\theta_i, \theta_{-i}), c_b^i(\theta_i, \theta_{-i})$$

- Feasibility:

$$c_a^1(\theta_1, \theta_2) + c_a^2(\theta_1, \theta_2) = 2e_a$$
$$c_b^1(\theta_1, \theta_2) + c_b^2(\theta_1, \theta_2) = 2e_b$$

- Social Planner, Mechanism Designer, Arbitrator, Principal: allocate resources across these two types efficiently
  Objective - utilitarian:

$$U^1 + U^2$$

  Could have other objectives as well
- Suppose everybody knows everything: planner and households
- Efficient to equate everyone's marginal utility of the same good:

$$\theta_1 u'(c_a^1(\theta_1, \theta_2) = \theta_2 u'(c_a^2(\theta_1, \theta_2))$$

$$u'(c_b^1(\theta_1, \theta_2) = u'(c_b^2(\theta_1, \theta_2))$$

- Consumption of bananas is equated; whoever has a higher taste for apples should consume more.
- Competitive equilibrium?
- Now suppose θ_i is privately known only by i.
- How do agents and the planner communicate?

- Assume for now they announce their type to the planner and he/she gives them allocations
- Would they truthfully announce? No. θ_L always wants to lie.
- Mechanism design: how do we design games between agents to achieve "efficiency"

1.3 Mechanism

- Need to figure out what is the best way to communicate
- Revelation principle: just focus on direct mechanisms
- Let's generalize:
  - N agents; K consumption goods: $c = (c_1, c_2, \cdots, c_K)$
  - Agents types, determine their prefs: $U^i(c) = u(c; \theta_i)$
  - $\theta = (\theta_1, \cdots, \theta_N) \in \Theta$ has some joint distribution $\mu(\theta)$
  - Aggregate endowments: $e = (e_1, \cdots, e_K)$;
  - Feasibility as always
$$\sum_{i=1}^{N} c_j^i = e_j$$
    set of feasible allocations $\mathcal{C} = \{(c^1, \cdots, c^N); c^i = (c_1^i, c_2^i, \cdots, c_K^i), \{c_j^i\}\text{: feasible}\}$
- What is a mechanism: (M, g)
  - M: messaging space
  - g: allocation rule
- Agent i send signal m_i (can only communicate with the planner); $m = (m_1, \cdots, m_N) \in M$
- allocation rule: $g : M \to \mathcal{C}$; $g(m) = (g^1(m), \cdots, g^N(m))$
- Direct mechanism: $M = \Theta$, $g(m) = c(\theta)$
- Each mechanism induces a game with asymmetric information
- Outcome induced by a mechanism: the Bayesian Nash Equilibrium of the game strategy profile:
$$m_*(\theta) = \left(m_*^1(\theta_1), \cdots, m_*^N(\theta_N)\right)$$
  such that

$$E\left[u\left(g^i\left(m_*^i(\theta_i), m_*^{-i}(\theta_{-i})\right), \theta_i\right)|\theta_i\right] \geq E\left[u\left(g^i\left(\hat{m}^i, m_*^{-i}(\theta_{-i})\right), \theta_i\right)|\theta_i\right] \forall \hat{m}^i, i$$

- Most common outcome concept; could also consider dominant strategy equilibrium
- A mechanism designer wants to find the best mechanism; complicated problem

1.4 Revelation Principle

Theorem 1. *An allocation that is an outcome of an arbitrary mechanism can be outcome of a direct mechanism where all agents reveal their types truthfully.*

Proof. Consider a mechanism (M, g) and an outcome m_*, it must satisfy

$$E\left[u\left(g^i\left(m_*^i(\theta_i), m_*^{-i}(\theta_{-i})\right), \theta_i\right)|\theta_i\right] \geq E\left[u\left(g^i\left(\hat{m}^i, m_*^{-i}(\theta_{-i})\right), \theta_i\right)|\theta_i\right] \forall \hat{m}^i, i$$

In particular

$$E\left[u\left(g^i\left(m_*^i(\theta_i), m_*^{-i}(\theta_{-i})\right), \theta_i\right)|\theta_i\right] \geq E\left[u\left(g^i\left(m_*^i(\hat{\theta}_i), m_*^{-i}(\theta_{-i})\right), \theta_i\right)|\theta_i\right], \forall \hat{\theta}_i, i$$

New mechanism

$$M = \Theta, c(\theta) = g(m_*(\theta))$$

Truth-telling is 'an' equilibrium. ∎

- Incentive compatible allocations, $c = (c^1, \cdots, c^N)$,

$$E\left[u\left(c^i\left(\theta_i \theta_{-i}\right)\right), \theta_i\right)|\theta_i\right] \geq E\left[u\left(c^i\left(\hat{\theta}_i, \theta_{-i}\right)\right), \theta_i\right)|\theta_i\right], \forall \hat{\theta}_i, i$$

- There could be other equilibria.
- Implementation theory: how do we ensure that it is the unique equilibrium? might cover it at the end.
- Most applied work; people usually don't announce their 'types'; decentralization with real world contracts – will see an example soon.
- Mechanism designer's problem

$$\max \sum_i \lambda^i E\left[u(c^i(\theta), \theta_i)\right]$$

subject to

$$\sum_i c^i_j(\theta) = e_j, \forall \theta \in \Theta, j = 1, \cdots, K$$

$$E\left[u\left(c^i\left(m^i_*(\theta_i), m^{-i}_*(\theta_{-i})\right), \theta_i\right)|\theta_i\right] \geq E\left[u\left(c^i\left(m^i_*(\hat{\theta}_i), m^{-i}_*(\theta_{-i})\right), \theta_i\right)|\theta_i\right], \forall \hat{\theta}_i, i$$

Continuum of agents

- Suppose there is a continuum of agents and shocks are i.i.d. across agents: no aggregate shocks
- $\Theta = \theta_1 < \cdots < \theta_N, \Pr(\theta_i) = \pi(\theta_i)$.
- Symmetric allocations: allocations only depend on individual types $c(\theta_i) = (c_1(\theta_i), c_2(\theta_i), \cdots, c_K(\theta_i))$
- feasible allocations:

$$\sum_i \pi(\theta_i) c_j(\theta_i) = e_j$$

- incentive compatible allocations:

$$u(c(\theta), \theta) \geq u(c(\hat{\theta}), \theta)$$

- Mechanism design problem:

$$\max \sum_i \lambda(\theta_i) u(c(\theta_i), \theta_i)$$

  subject to Feasbility and incentive compatibility

- Go through a two type simple example

2 First Application - Mirrlees

- If you want to read about this stuff take a look at Salanie's book on Economics of Taxation; chapter 4, also Mirrlees(1971), Diamond(1998), and Saez(2001)
- Last week: two types example; marginal tax rate zero at the top, positive at the bottom
- How general?

- Here continuous types; Recall problem

$$\max \int \left[u\left(c\left(\theta\right)\right) - v\left(\frac{y\left(\theta\right)}{\theta}\right)\right] dH\left(\theta\right)$$

subject to

$$\begin{aligned} \int c\left(\theta\right) dF\left(\theta\right) + G &\leq \int y\left(\theta\right) dF\left(\theta\right) \\ u\left(c\left(\theta\right)\right) - v\left(y\left(\theta\right)/\theta\right) &\geq u\left(c\left(\hat{\theta}\right)\right) - v\left(y\left(\hat{\theta}\right)/\theta\right) \end{aligned}$$

- Remember $H\left(\theta\right)$ cumulative welfare weight; $H\left(\theta\right) \geq F\left(\theta\right)$ with derivative $h\left(\theta\right)$. what is the interpretation of $h\left(\theta\right)$?

- Notice: the constraint set can be rewritten as a maximization problem

$$u\left(c\left(\theta\right)\right) - v\left(y\left(\theta\right)/\theta\right) = \max_{\hat{\theta}} u\left(c\left(\hat{\theta}\right)\right) - v\left(y\left(\hat{\theta}\right)/\theta\right)$$

- One way to simplify the constraint: replace it with its first and second order conditions!!

  FOC:

$$\left\{ u'\left(c\left(\hat{\theta}\right)\right) c'\left(\hat{\theta}\right) - \frac{1}{\theta} y'\left(\hat{\theta}\right) v'\left(y\left(\hat{\theta}\right)/\theta\right) \right\}\Big|_{\hat{\theta}=\theta} = 0$$

  or

$$u'\left(c\left(\theta\right)\right) c'\left(\theta\right) - \frac{1}{\theta} y'\left(\theta\right) v'\left(y\left(\theta\right)/\theta\right) = 0$$

  SOC: After cumbersome algebra:

$$y'\left(\theta\right) \geq 0$$

  One way to see this:

$$\begin{aligned} u\left(c\left(\theta\right)\right) - v\left(y\left(\theta\right)/\theta\right) &\geq u\left(c\left(\hat{\theta}\right)\right) - v\left(y\left(\hat{\theta}\right)/\theta\right) \\ u\left(c\left(\hat{\theta}\right)\right) - v\left(y\left(\hat{\theta}\right)/\hat{\theta}\right) &\geq u\left(c\left(\theta\right)\right) - v\left(y\left(\theta\right)/\hat{\theta}\right) \end{aligned}$$

  Suppse $\theta > \hat{\theta}$. Add them up, c's cancel and:

$$v\left(y\left(\theta\right)/\hat{\theta}\right) - v\left(y\left(\theta\right)/\theta\right) \geq v\left(y\left(\hat{\theta}\right)/\hat{\theta}\right) - v\left(y\left(\hat{\theta}\right)/\theta\right)$$

  What does this inequality look like

$$h\left(1/\hat{\theta}, y\left(\theta\right)\right) - h\left(1/\theta, y\left(\theta\right)\right) \geq h\left(1/\hat{\theta}, y\left(\hat{\theta}\right)\right) - h\left(1/\theta, y\left(\hat{\theta}\right)\right)$$

  where $h\left(x, y\right) = v\left(xy\right)$ and hence $h_x = yv'\left(xy\right)$.

- Can write this as

$$\int_{1/\theta}^{1/\hat{\theta}} h_x(x, y(\theta))\, dx \geq \int_{1/\theta}^{1/\hat{\theta}} h_x\left(x, y\left(\hat{\theta}\right)\right) dx$$

or

$$y(\theta) \int_{1/\theta}^{1/\hat{\theta}} v'(xy(\theta))\, dx \geq y\left(\hat{\theta}\right) \int_{1/\theta}^{1/\hat{\theta}} v'\left(xy\left(\hat{\theta}\right)\right) dx$$

Now because v is convex, it must be that $y(\theta) \geq y\left(\hat{\theta}\right)$. To see this, suppose not, $y(\theta) < y\left(\hat{\theta}\right)$.Then

$$v'(xy(\theta)) < v'\left(xy\left(\hat{\theta}\right)\right)$$

and this clearly violates the above inequality.

- So relaxed problem

$$\max \int \left[u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right)\right] dH(\theta)$$

subject to

$$\begin{aligned}
\int c(\theta)\, dF(\theta) + G &\leq \int y(\theta)\, dF(\theta) \\
u'(c(\theta))\, c'(\theta) - \frac{1}{\theta} y'(\theta)\, v'(y(\theta)/\theta) &= 0 \\
y'(\theta) &\geq 0
\end{aligned}$$

- We call the second constraint local incentive compatibility. You can show that when local IC holds and $y'(\theta) \geq 0$, then IC holds. For proof see Fudenberg and Tirole, chapter 7.

- Now this problem is still ugly. A trick a la Mirrlees: this FOC is equivalent to an Envelop condition. How? Let

$$U(\theta) = u(c(\theta)) - v(y(\theta)/\theta)$$

Then

$$U(\theta) = \max_{\hat{\theta}} u\left(c\left(\hat{\theta}\right)\right) - v\left(y\left(\hat{\theta}\right)/\theta\right)$$

- Envelope:

$$\begin{aligned} U'(\theta) &= \frac{\partial}{\partial \theta} u\left(c\left(\hat{\theta}\right)\right) - v\left(y\left(\hat{\theta}\right)/\theta\right)\big|_{\hat{\theta}=\theta} \\ &= \left.\frac{y\left(\hat{\theta}\right)}{\theta^2} v'\left(y\left(\hat{\theta}\right)/\theta\right)\right|_{\hat{\theta}=\theta} \\ &= \frac{y(\theta)}{\theta^2} v'\left(y(\theta)/\theta\right) \end{aligned}$$

  Rewrite the problem with this

$$\max \int U(\theta)\, dH(\theta)$$

  subject to

$$\begin{aligned} u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right) &= U(\theta) \\ \int c(\theta)\, dF(\theta) + G &\leq \int y(\theta)\, dF(\theta) \\ U'(\theta) &= \frac{y(\theta)}{\theta^2} v'\left(y(\theta)/\theta\right) \\ y'(\theta) &\geq 0 \end{aligned}$$

- Common technique in Mechanism Design problem: Let's for now ignore the last constraint. Later we have to check if it is satisfied. If violated, it means that the original allocation has bunching of types; people have the same allocations. There is a procedure called ironing; see Myerson's 'Optimal Auction Design'.

- Looks like a standard Hamiltonian. We can use the standard techniques. If you wanna learn more about calculus of variation, look at Luenberger, 'Optimization by Vector Space Methods'. Let me quickly go over it; we can write the Lagrangian

$$\begin{aligned} L &= \int U(\theta) h(\theta)\, d\theta + \int \gamma(\theta) \left[u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right) - U(\theta)\right] d\theta \\ &\quad + \lambda \left[\int y(\theta) f(\theta)\, d\theta - \int c(\theta) f(\theta)\, d\theta - G\right] \\ &\quad + \int \mu(\theta) \left[U'(\theta) - \frac{y(\theta)}{\theta^2} v'\left(y(\theta)/\theta\right)\right] d\theta \end{aligned}$$

  If U' wasn't there, we could simply take 'derivative'!! So let's try and get rid of it!! We can use integration by part to do just that

$$\begin{aligned}\int_{\underline{\theta}}^{\bar{\theta}} \mu(\theta)U'(\theta)d\theta &= \int \mu(\theta)\, dU(\theta) \\ &= \mu(\bar{\theta})\, U(\bar{\theta}) - \mu(\underline{\theta})\, U(\underline{\theta}) - \int \mu'(\theta)\, U(\theta)\, d\theta\end{aligned}$$

- Now, we can take FOCs, i.e., do a calculus of variations argument

$$\begin{aligned}h - \gamma - \mu' &= 0 \\ \gamma u'(c) - \lambda f &= 0 \\ -\gamma \frac{1}{\theta} v'\left(\frac{y}{\theta}\right) + \lambda f - \mu\left[\frac{1}{\theta^2} v'\left(\frac{y}{\theta}\right) + \frac{y}{\theta^3} v''\left(\frac{y}{\theta}\right)\right] &= 0 \\ \mu(\bar{\theta}) = \mu(\underline{\theta}) &= 0\end{aligned}$$

Let's do some algebra

$$\begin{aligned}\mu &= \int_{\theta}^{\bar{\theta}} (\gamma - h)\, d\hat{\theta} \\ &= \int_{\theta}^{\bar{\theta}} \left(\frac{1}{u'(c)} \lambda f - h\right) d\hat{\theta}\end{aligned}$$

- Then

$$\lambda f - \frac{\lambda f}{u'(c)} \frac{1}{\theta} v'\left(\frac{y}{\theta}\right) = \left[\frac{1}{\theta^2} v'\left(\frac{y}{\theta}\right) + \frac{y}{\theta^3} v''\left(\frac{y}{\theta}\right)\right] \int_{\theta}^{\bar{\theta}} \left(\frac{1}{u'(c)} \lambda f - h\right) d\hat{\theta}$$

divide by $\frac{1}{\theta} v'(y/\theta)$,

$$\frac{1}{1/\theta v'(y/\theta)} - \frac{1}{u'(c)} = \frac{1}{\theta f}\left[1 + \frac{y/\theta v''(y/\theta)}{v'(y/\theta)}\right]\left[\int_{\theta}^{\bar{\theta}} \frac{1}{u'(c(\hat{\theta}))} \frac{dF(\hat{\theta})}{1-F} - \frac{1-H}{\lambda(1-F)}\right]$$

$$\begin{aligned}\frac{1}{u'(c)(1-\tau(\theta))} - \frac{1}{u'(c)} &= \frac{1-F}{\theta f}\left[1 + \frac{y/\theta v''(y/\theta)}{v'(y/\theta)}\right]\left[\int_{\theta}^{\bar{\theta}} \frac{1}{u'(c(\hat{\theta}))} \frac{dF(\hat{\theta})}{1-F} - \frac{1-H}{\lambda(1-F)}\right] \\ \frac{\tau}{1-\tau} &= \frac{1-F}{\theta f}\left[1 + \frac{y/\theta v''(y/\theta)}{v'(y/\theta)}\right]\left[\int_{\theta}^{\bar{\theta}} \frac{u'(c(\theta))}{u'(c(\hat{\theta}))} \frac{dF(\hat{\theta})}{1-F} - \frac{(1-H)\, u'(c)}{\lambda(1-F)}\right]\end{aligned}$$

Also

So called Mirrlees-Diamond-Saez formula. Can be useful and misleading – A function of a bunch of endogenous stuff!

Loosely speaking, it says that marginal tax rates are a function of labor supply elasticity, tail of the distribution and how redistributive the objective is. Let's go through them and see why

- The first term $\frac{1-F}{\theta f}$ is the tail of the productivity distribution. If the productivity distribution looks like a lognormal distribution or bounded above this term converges to zero; taxes should be zero at the top. If productivity distribution looks like a Pareto distribution at the top – has a fat tail, then this term is positive.
- The second term looks a lot like an elasticity. In fact it is one plus the Frisch elasticity of labor supply. So the more elastic is labor supply, the lower the taxes.
- The last term is a complicated term. Somehow it measures how unequal is consumption allocated as well as how redistribitive is the planner's motive. Suppose that $H(\theta)$ goes up and nothing else changes, then it implies that the tax rates should go up; a planner that would like to do more redistribution would like to increase taxes.

• What happens to the top tax rate if the distribution of θ is unbounded. Suppose that H converges to 1 really fast!!!, then

$$\lim \frac{\tau}{1-\tau} = \lim \frac{1-F}{\theta f} \lim (1 + 1/\varepsilon)$$

• Before providing some numbers. An exactly solved case. suppose that $u(c) = c$. Then:

$$\frac{\tau(\theta)}{1-\tau(\theta)} = \left(1 + \frac{1}{\varepsilon}\right) \frac{G(\theta) - F(\theta)}{\theta f(\theta)}$$

a function of fundamentals. Can perform comparative statics.

• We can calculate taxes at the top from micro data. What's the shape of the income distribution. Can approximate it with Pareto. See Saez's pictures.

Suppose that

$$1 - F = C\theta^{-a}$$

Then

$$f = aC\theta^{-a-1}$$

So

$$\frac{1-F}{\theta f} = \frac{1}{a}$$

• Micro estimates of labor supply elasticity: .2-.5. Need to know a.

- From data we know the pareto tail for income (dont really see θ). a from income distribution is not the same as a above. But we know current taxes and can back out θ from that

  intratemporal Euler equation

$$(1-\tau_{auctual})\,\theta = \psi l^{1/\varepsilon}$$

  Hence roughly at the top

$$y = \left[\frac{1-\tau_{auctual}}{\psi}\right]^{\varepsilon}\theta^{\varepsilon+1}$$

  If a_{income}, then a_θ can be calculated from

$$\begin{aligned}
1-\Pr\left(\theta \leq \hat{\theta}\right) &= 1-\Pr\left(\left[\frac{1-\tau_{auctual}}{\psi}\right]^{\varepsilon}\theta^{\varepsilon+1} \leq \left[\frac{1-\tau_{auctual}}{\psi}\right]^{\varepsilon}\hat{\theta}^{\varepsilon+1}\right) \\
&= C\left[\frac{1-\tau_{auctual}}{\psi}\right]^{-\varepsilon a_{income}}\hat{\theta}^{-(\varepsilon+1)a_{income}}
\end{aligned}$$

  So $a_\theta = (1+\varepsilon)\,a_{income}$.So

$$\begin{aligned}
\frac{\tau}{1-\tau} &= \frac{1}{(1+\varepsilon)\,a_{income}}\frac{1+\varepsilon}{\varepsilon} \\
&= \frac{1}{\varepsilon a_{income}}
\end{aligned}$$

- Then $\frac{\tau}{1-\tau}$ should be between 2.5 and 1. So τ should be somewhere between .5 and .71. Very high numbers!!!

- With income effect, the numbers would be higher. Now let's look at Saez's numbers.

- What is wrong with this

  - Conceptual level: infinitely skilled people!!! However, in many practical cases, zero-tax-at-the-top is local. How do we know the only reason people are unequal is because of skills?
  - suppose we take this model to be stand in for what happens in the world. Then elasticity of labor supply has to be way higher: human capital accumulation, career concerns, extensive margin, etc.
  - Hopefully, when we get to dynamics, we can talk about these things. There is not a lot that has been done. and there is a lot more to be done!

3 Second Application - Diamond and Dybvig

- A dynamic model but we can analyze the problem using our tools.
- What is the role of banks?
- Maturity transformation:
  - People need to eat everyday, i.e., they are hit by liquidity shocks
  - Production takes time, i.e., physical capital is illiquid.
  - Need contracts that insure people against liquidity shocks
  - One way to do this is through banks
- Let's try to formalize this
- Time: $t = 0, 1, 2$; Continuum of households, eat at $t = 1, 2$:

$$(1-\theta)u(c_1) + \theta u(c_1 + c_2)$$

- $\theta \in \{0, 1\}$:types realized at $t = 1$. $\pi = \Pr(\theta = 0)$.
- Utility function

$$-\frac{cu''(c)}{u'(c)} > 1$$

- Technology:
  - Short term investment: can invest at $t = 0, 1$ will give a return of 1 at $t + 1$.(liquid asset)
  - Long term investment: invest at $t = 0$, returns R rate of return at $t = 2$.(illiquid asset); if liquidated at $t = 1$, pays back its original investment

$$R > 1$$

- Households: endowed with e unit at $t = 0$.

- Allocations: $c_t(\theta), t = 1, 2$, x:investment in liquid asset,y: investment in illiquid asset

- Feasibility:

$$
\begin{aligned}
x + y &= e \\
\pi c_1(0) + (1-\pi) c_1(1) &= x \\
\pi c_2(0) + (1-\pi) c_2(1) &= Ry
\end{aligned}
$$

or

$$
\pi \left[c_1(0) + \frac{1}{R} c_2(0) \right] + (1-\pi) \left[c_1(1) + \frac{1}{R} c_2(1) \right] \leq 1
$$

- Incentive constraint:

$$
u(c_1(0)) \geq u(c_1(1))
$$

$$
u(c_1(1) + c_2(1)) \geq u(c_1(0) + c_2(0))
$$

- Mechanism Design Problem:

$$
\max U
$$

subject to feasibility, incentive compatibility.

- Properties of optimal allocations:

$$
\begin{aligned}
c_2(0) &= c_1(1) = 0 \\
u'(c_1(0)) &= R u'(c_2(1))
\end{aligned}
$$

- Since risk aversion is higher than 1, $1 < c_1(0) < c_2(1) < R$. There is some insurance going on. Boring mechanism design problem!!! i.e., incentive constraints are non-binding. What happens if we have other types?

3.1 Competitive Equilibrium

- Suppose that households can only trade the long and short asset; budget constraint at date 0.

$$
x^i + y^i = 1
$$

- If they happen to become impatient, they can sell their claims to long asset to the patient guys in exchange for claims to the short asset.

$$
c_1(0) = x^i + P y^i
$$

Then the consumption of patient guys is

$$c_2(1) = \left(x^i/P + y^i\right) R$$

- Definition of competitive equilibrium: $\{x^i, y^i\}_{i\in[0,1]}$ such that x^i, y^i solves

$$\max_{x,y} \pi u(c_1(0)) + (1-\pi) u(c_2(1))$$

subject to

$$\begin{aligned} & x + y = 1 \\ & c_1(0) = x + Py \\ & c_2(1) = (y + x/P) R = (x + Py)\frac{R}{P} \end{aligned}$$

- Market clearing:

$$\begin{aligned} \text{at } t &= 1, \pi \int c_1^i(0)\, di = \int x^i di \\ \text{at } t &= 0, (1-\pi)\int c_2^i(1) = R\int y^i di \end{aligned}$$

- In equilibrium $P = 1$. Why?
  - If $P > 1$, then the solution to the above problem requires $x = 0$. violates market clearing!
  - If $P < 1$, then the solution to the above problem requires $y = 0$. violates market clearing!
- with $P = 1$, $c_1(0) = 1$, $c_2(1) = R$. Inefficient. Reason: market incompleteness; no insurance against θ. they want to trade contingent claim on θ. But θ is private information, so they can potentially misreport(Will see later that competitive markets with private information are not necessarily inefficient; see Prescott and Townsend(1982))

3.2 Bank Run

- Let's try some other contracts
- Suppose there is a liquidation technology. We can decide whether to liquidate the long asset at $t = 1$ and get return 1.

- Banks offer a deposit contract, deposit your money at $t = 0$; if withdraw at $t = 1$, then return is r_1 if the fraction of people withdrawing is less then $1/r_1$. If this fraction is higher, then return is zero.

$$V_1(f_j, r_1) = \begin{cases} r_1 & f_j \leq 1/r_1 \\ 0 & f_j > 1/r_1 \end{cases}$$

$$V_2(f, r_1) = \max\{R(1 - fr_1)/(1 - f), 0\}$$

- A game between patient and impatient guys. Now, what happens in the equilibrium of this game: two equilibria
- No Run: impatient guys are boring. if patient guys think other patient guys will not run

$$\begin{aligned} \text{No Run} &: \quad u\left(\frac{R(1-\pi r_1)}{1-\pi}\right) \\ \text{Run} &: \quad u(r_1) \end{aligned}$$

  He wont run as long as

$$r_1 \leq \frac{R(1-\pi r_1)}{1-\pi}$$

  Set $r_1 = c_1(0)$. Done.

- Run equilibrium: if patient guys think other patient guys will run:
-

$$\begin{aligned} \text{No Run} &: \quad u(0) \\ \text{Run} &: \quad 1/r_1 u(r_1) + (1 - 1/r_1) u(0) \end{aligned}$$

  If $r_1 > 1$, there is an equilibrium with Bank Runs!

- Suspension of convertability can potentially solve this
- If we add aggregate shocks to π, then still a problem
- Summary: Deposit contracts could lead to "fragility"!!!! Diamond and Dybvig: We need deposit insurance; but do we?
- Suppose we just ask them and dont care about sequential service. Also suppose that the number of households is finite, I; Number of patient guys: stochastic
- Planner's problem

$$\max \sum \Pr(\omega)\left[\alpha(\omega) u(c_1(0,\omega)) + (1-\alpha(\omega)) u(c_2(1,\omega))\right]$$

subject to

$$(I-\alpha(\omega))\, c_1(0,\omega)+\frac{1}{R}\alpha(\omega)\, c_2(1,\omega)=I$$

where $\omega=(\theta_1,\cdots,\theta_I)$.

- FOCs

$$u'(c_1(0,\omega))=Ru'(c_2(1,\omega))$$

Can represent the state of the world by α: $c_1(0,\alpha)$, $c_2(1,\alpha)$.

- Suppose that $u(c)=c^{1-\sigma}/(1-\sigma)$ with $\sigma>1$

$$c_1(0,\alpha)^{-\sigma}=Rc_2(1,\alpha)^{-\sigma}$$

$$c_2(1,\alpha)=R^{\frac{1}{\sigma}}c_1(0,\alpha)$$

$$\begin{aligned}
c_1(0,\alpha) &= \frac{I}{\alpha+R^{1/\sigma-1}(I-\alpha)}\\
&= \frac{I}{\alpha+R^{1/\sigma-1}(I-\alpha)}\\
c_2(1,\alpha) &= \frac{IR^{1/\sigma}}{\alpha\left(1-R^{1/\sigma}\right)+IR^{1/\sigma-1}}
\end{aligned}$$

Since $R>1$, $c_2(1,\alpha)>c_1(0,\alpha)$. Also $c_1(0,\alpha)>c_1(0,\alpha-1)$.

- Doesn't this mean that truth-telling is a dominant strategy? No Bank Runs!!!!
- Green-Lin(2003, JET)(gerenalized by Andolfatto, Nosal, Wallace(2007, JET)) show that this argument also works with sequential service but it's a bit more complicated:
  - Intuition: The last patient trader does not want to lie(Given the remaining resources with the bank, the planner always gives more to the patient guy). Using a backward induction argument the second to last trader doesn't want to lie either. Nobody wants to lie! No Bank Run.
- Ennis and Kiester(2009, JET): if you allow for correlated types the truthfull revelation game might have other equilibria.
- Nosal(2011); Cavalcanti and Monteiro(2011): there are indirect mechanisms that have unique equilibrium and implement the efficient outcome.
- Bank Runs??!!
- DD: Bank run one issue that can be a result of maturity mismatch.

3.3 Non-Exclusive Contracts and The Role of Banks: Jacklin/Farhi-Golosov-Tsyvinski

- Another issue pointed by Jacklin elaborated by Farhi, Golosov, and Tsyvinski: The above analysis assumes exclusivity; households cannot contract with each other and with other banks. Can potentially undo the insurance provided by the intermediaries.

- To see this consider the following alternative private market arrangement:

  - There is a lot of banks; all of them are large, i.e., can service the whole market.
  - They offer contracts at $t = 0$ and commit to them
  - Households choose the contract with highest utility
  - Before talking about non-exclusivity, let's focus on exclusive contracts: Assume that banks can control how much people consume.
  - Equilibrium is given by $\{c_t(\theta)\}, \bar{U}$, where $\bar{U}$ is the value of equilibrium contract to households and $c_t(\theta)$ solves the following maximization problem by the intermediary(they take $\bar{U}$ as given)

$$\max d_1 + \frac{1}{R} d_2$$

subject to

$$\begin{aligned} x + y &= 1 \\ d_1 + \pi c_1(0) + (1-\pi) c_1(1) &= x \\ d_2 + \pi c_2(0) + (1-\pi) c_2(1) &= Ry \\ u(c_1(0)) &\geq u(c_1(1)) \\ u(c_1(1) + c_2(1)) &\geq u(c_1(0) + c_2(0)) \\ \pi u(c_1(0)) + (1-\pi) u(c_1(1) + c_2(1)) &\geq \bar{U} \end{aligned}$$

and markets clear.

- Let's focus on the truth-telling equilibrium so do not have to deal with bank runs.

- Now, we want to show that the equilibrium outcome is efficient: when contracts are exclusive competitive markets are providing the efficient amount of insurance that a planner offers.

- Obviously profits in equilibrium have to be zero. If positive, some other intermediary can come in offer a little more generous contract, make positive profits and service the whole market.

- To show this first notice that as usual $c_2(0) = c_1(1) = 0$. Then the IC's can be written as

$$\begin{aligned} u(c_1(0)) &\geq u(0) \\ u(c_2(1)) &\geq u(c_1(0)) \end{aligned}$$

If we didn't have the incentive constraints in there, we could use concavity properties to rewrite the bank's problem as its dual and then bank's problem would coincide with planner's problem. So how do we deal with incentive constraints(potentially they can make the problem non-covex because a concave function is appearing on the RHS of the IC's). One way to deal with this in this environment is to rename stuff: Let

$$\begin{aligned} \hat{u}_1(0) &= u(c_1(0)) \\ \hat{u}_2(1) &= u(c_2(1)) \end{aligned}$$

Then, the bank's problem can be written as

$$\max 1 - \pi C(\hat{u}_1(0)) - (1-\pi)\frac{1}{R}C(\hat{u}_2(1))$$

subject to

$$\begin{aligned} \hat{u}_1(0) &\geq u(0) \\ \hat{u}_1(2) &\geq \hat{u}_1(1) \\ \pi\hat{u}_1(0) + (1-\pi)\hat{u}_2(1) &\geq \bar{U} \end{aligned}$$

where $C(u)$ is the inverse of the utility function $u(c)$. Now the constraint set is linear and the objective is strictly concave – C is strictly convex and $-C(u)$ is strictly concave.

- So we can write down the dual of the above problem and use the fact that profits in equilibrium are zero

$$\bar{U} = \max \pi\hat{u}_1(0) + (1-\pi)\hat{u}_2(1)$$

subject to

$$\begin{aligned} \hat{u}_1(0) &\geq u(0) \\ \hat{u}_1(2) &\geq \hat{u}_1(1) \\ 1 - \pi C(\hat{u}_1(0)) - (1-\pi)\frac{1}{R}C(\hat{u}_2(1)) &\geq 0 \end{aligned}$$

Now you can see that this coincides with the planning proble before. So the equilibrium must be efficient. This is a proof similar to the one used by Prescott and

Townsend(1984, Econometrica). They show that if contracts are signed before any uncertainty is realized and people have access to lotteries (This is a trick to convexify the constraint set – in the above proof, we didn't really need it given the special structure of this problem), then Competitive Equilibrium is constrained efficient.

- Now let's try to see what happens if we have non-exclusive contracts

Alternative[2] Competitive Market:

- Let's assume that household's have access to trading market where they can trade with each other but the intermediaries cannot see what they do in those markets, i.e., hidden trading markets. There is an interest rate $\hat{R}$ in this market. So if they face a contract $\{c_t(\theta)\}$ from an intermediary, this is the maximization problem they solve

$$\max_{\hat{\theta}, s, x_t} (1-\theta) u(x_1) + \theta u(x_2)$$

subject to

$$\begin{aligned} x_1 + s &= c_1(\hat{\theta}) \\ x_2 &= \hat{R}s + c_2(\hat{\theta}) \end{aligned}$$

As it can be seen, they can decide to announce their type to the bank, take the allocations from the bank, and trade in the hidden trade market. Call the value associated with the above $V\left(\{c_t(\theta)\}, \theta, \hat{R}\right)$. We also call the optimal choice of reporting by $\hat{\theta}\left(\{c_t(\theta)\}, \hat{R}, \theta\right)$ and consumption by $x_t\left(\{c_t(\theta)\}, \hat{R}, \theta\right)$.

- Major assumption: there is full commitment in the 'Hidden Trade' Market.

- Equilibrium Definition: equilibrium is described by $\{x_t(\theta), c_t(\theta)\}, \bar{U}, \hat{R}$ such that

  - The intermediary solves the following problem

$$\max d_0 + \frac{1}{R} d_1$$

such that

$$\begin{aligned} & x + y = 1 \\ & d_1 + \pi c_1(0) + (1-\pi) c_1(1) = x \\ & d_2 + \pi c_2(0) + (1-\pi) c_2(1) = Ry \\ & \theta = \hat{\theta}\left(\{c_t(\theta)\}, \hat{R}, \theta\right) \\ & \pi V\left(\{c_t(\theta)\}, \hat{R}, 0\right) + (1-\pi) V\left(\{c_t(\theta)\}, \hat{R}, 1\right) \geq \bar{U} \end{aligned}$$

– Hidden Trade Market has to clear:

$$\pi x_t\left(\{c_t(\theta)\},\hat{R},0\right)+(1-\pi)\,x_t\left(\{c_t(\theta)\},\hat{R},1\right)\leq$$
$$\pi c_t\left(\hat{\theta}\left(\{c_t(\theta)\},\hat{R},0\right)\right)+(1-\pi)\,c_t\left(\hat{\theta}\left(\{c_t(\theta)\},\hat{R},1\right)\right)$$

- First Result: IC implies that

$$c_1(0)+\frac{1}{\hat{R}}c_2(0)=c_1(1)+\frac{1}{\hat{R}}c_2(1)$$

  Note that since consumers can do smoothing on their own in the hidden trading market, all they care about is the present value of consumption bundles offered(where PV is calculted using hidden trade market prices, $\hat{R}$). So they always choose the bundle with the highest present value. IC means that the present values have to be equal.

- Now, given the fact that banks make zero profits in equilibrium and a similar duality result as before, competitive equilibrium is a solution to the following maximization problem

$$\max \pi V\left(\{c_t(\theta)\},\hat{R},0\right)+(1-\pi)\,V\left(\{c_t(\theta)\},\hat{R},1\right)$$

  subject to

$$c_1(0)+\frac{1}{\hat{R}}c_2(0)=c_1(1)+\frac{1}{\hat{R}}c_2(1)$$
$$\pi\left(c_1(0)+\frac{1}{R}c_2(0)\right)+(1-\pi)\left(c_1(1)+\frac{1}{R}c_2(1)\right)\leq 1$$

- Second Result: $\hat{R}=R$. Suppose that $\hat{R}<R$, then clearly in the solution to the above problem $c_1(0)=c_1(1)=0$.Why? because if not you can decrease $c_1(0)$ by ε, and increase $c_2(0)$ by $\left(\pi R+(1-\pi)\hat{R}\right)\varepsilon$ and $c_2(1)$ by $\pi\varepsilon\left(R-\hat{R}\right)$. This causes the present values to increase equally(hence V increases for both types) and feasibility to be satisfied. In other words, it is optimal for the intermediary to invest in the long asset only. However, this cannot be an equilibrium because, when $c_1(0)=c_1(1)=0$, impatient types cannot consume anything given market clearing in the hidden trading market. Similarly we can show that $R<\hat{R}$ cannot be an equilibrium either.

- When $R=\hat{R}$, the equilibrium outcome is clearly $c_1(0)=1$, $c_2(1)=R$.

- Notice that this outcome is identical to the competitive equilibrium we discussed above. However, there market incompleteness was exogenous – we assumed some assets do not exist, while here market incompleteness is endogenous – Fundamental frictions result in market incompleteness.

- So is the above allocation constraint efficient? Clearly it is not efficient with respect to the original planning problem. But we have to define what constrainted efficient means with hidden trade.

Alternative[2]Constrained Efficient Allocation.

- We will assume that a planner is subject to the same hidden trading constraint – people can trade in the hidden trade market behind planner's back. However, by choosing allocations, the planner can affect the interest rate in the hidden trade market indirectly. So, here is the planning problem

$$\max_{c_t(\theta)} \pi U\left(c_1\left(0\right), c_2\left(0\right), 0\right) + \left(1-\pi\right) U\left(c_1\left(1\right), c_2\left(1\right), 1\right)$$

subject to

$$\pi\left(c_1\left(0\right) + \frac{1}{R}c_2\left(0\right)\right) + \left(1-\pi\right)\left(c_1\left(1\right) + \frac{1}{R}c_2\left(1\right)\right) \leq 1$$
$$U\left(c_1\left(\theta\right), c_2\left(\theta\right); \theta\right) \geq V\left(\left\{c_t\left(\theta\right)\right\}, \hat{R}, \theta\right)$$
$$\hat{R} = \text{Eqilibiurm Int. Rate in HT Mkt}$$

- First note that similar to before, we must have

$$c_1\left(\theta\right) + \frac{1}{\hat{R}}c_2\left(\theta\right) = c_1\left(\theta'\right) + \frac{1}{\hat{R}}c_2\left(\theta'\right)$$

If not, the allocation would not be incentive compatible. Call the above present value I. Define, the following

$$\hat{V}\left(I, \hat{R}, \theta\right) = \max U\left(x_1, x_2; \theta\right)$$

subject to

$$x_1 + \frac{1}{\hat{R}}x_2 = I$$

Call the policy functions $x_1\left(I, \hat{R}, \theta\right), x_2\left(I, \hat{R}, \theta\right)$. Then it is easy to show that the planner's problem is equivalent to

$$\max_{I, \hat{R}} \pi \hat{V}\left(I, \hat{R}, 0\right) + \left(1-\pi\right)\hat{V}\left(I, \hat{R}, 1\right)$$

subject to

$$\pi\left(x_1\left(I, \hat{R}, 0\right) + \frac{1}{R}x_2\left(I, \hat{R}, 0\right)\right) + \left(1-\pi\right)\left(x_1\left(I, \hat{R}, 1\right) + \frac{1}{R}x_2\left(I, \hat{R}, 1\right)\right) \leq 1$$

- Note that if there exists I and $\hat{R}$ such that

$$
\begin{aligned}
x_1\left(I,\hat{R},0\right) &= c_1^*\left(0\right), x_2\left(I,\hat{R},0\right)=0 \\
x_2\left(I,\hat{R},1\right) &= c_2^*\left(1\right), x_1\left(I,\hat{R},1\right)=0
\end{aligned}
$$

where $c_1^*\left(0\right)$ and $c_2^*\left(1\right)$ is the solution to the original planning problem, then this must be optimal because the above problem is obtained from adding extra constraints on the original planning problem.

- Obviously $I = c_1^*\left(0\right)$ and $\hat{R} = \frac{c_2^*(1)}{c_1^*(0)}$. To see that this is the case, we just have to show that $\hat{R} > 1$, which is clear from the properties of the allocation described above. Moreover, we can see that $\hat{R} < R$. This is because risk aversion is bigger than 1.

$$u'\left(I\right) = Ru'\left(\hat{R}I\right)$$

$$\log u'\left(\hat{R}I\right) - \log u'\left(I\right) + \log R = 0$$

Note that

$$\frac{d}{dx}\log u'\left(x\right) = \frac{u''\left(x\right)}{u'\left(x\right)}$$

So

$$\log u'\left(\hat{R}I\right) - \log u'\left(I\right) = \int_I^{\hat{R}I} \frac{u''\left(x\right)}{u'\left(x\right)}dx$$

Since risk aversion is bigger than 1,

$$-\frac{u''\left(x\right)x}{u'\left(x\right)} > 1$$

Hence,

$$\frac{u''\left(x\right)}{u'\left(x\right)} < -\frac{1}{x}$$

So, we must have

$$
\begin{aligned}
\log u'\left(\hat{R}I\right) - \log u'\left(I\right) &= -\log R \\
&< -\int_I^{\hat{R}I}\frac{1}{x}dx = \log I - \log\left(\hat{R}I\right) = -\log\hat{R}
\end{aligned}
$$

Hence, $1 < \hat{R} < R$. So the planner would achieve the original efficient allocation.

- Note: this is not in general true. The Alternative[2] Constrained Efficient Allocation does not usually coincide with the Constrained Efficient Allocation.

- Why this inefficiency: As the above proof shows, the planner essentially affects the interest rate in the hidden trading market. This interest rate affect agents' incentives to lie. In particular, loweing interest rate $\hat{R}$ benefits impatient agents since it increases their income.

- So how can we implement this desired outcome? FGT show that a simple policy when the government can see banks' portfolios – or the banking sector's protfolio as a whole; there is no difference.

- Now consider a lower bound imposed by the government on banks' investment in the liquid/short asset and suppose this bound is given by πI^* where I^* is the solution to the above planning problem. It is clear from before that this would achieve the constrained efficient allocation. When $\hat{R} < R$, the banks want to invest everything in the long asset. The liquidity floor prevents that from happening.

4 Dynamic Mechanism Design

- Let's go back to our original example with taste shocks.

- Prefs:

$$\theta u\left(c_a\right) + u\left(c_b\right)$$

As we saw in the previous example, we can just call consumption across at each time a different good, i.e., we can call consumption in the first period apples and consumption in the second period bananas.

- What is different? Information usually arrives gradually.

- Let's try to formulate this(Based on Atkeson-Lucas, 1992). Consider a taste shock economy with preferences of the form

$$\sum_{t=0}^{T} \beta^t \theta_t u\left(c_t\right)$$

where $\theta_t \in \Theta$ is a Markov Process of order 1 with transition transition probabilities $\pi_t\left(\theta_{t+1}|\theta_t\right)$. Later we restrict attention to i.i.d. $T \in \mathbb{N} \cup \{\infty\}$.

- Continuum of households; All endowed with e unit of the consumption good in each period.

- Allocations: $c_t\left(\theta^t\right)$. Feasbility

$$\int_{\Theta^t} c_t\left(\theta^t\right) d\mu\left(\theta^t\right) \leq e$$

- Optimal risk sharing with full info

$$\beta^t \theta_t u'(c_t) = \lambda_t$$

for all t. λ_t : multiplier on feasibility at t.

- Now suppose θ_t is private information; Since there is commitment in this environment, revelation principle applies and we can focus on direct mechanisms. An allocation is incentive compatible, if

$$\sum_{t=0}^{T} \beta^t \int_{\Theta^t} \theta_t u\left(c_t\left(\theta^t\right)\right) d\mu_t\left(\theta^t\right) \geq \sum_{t=0}^{T} \beta^t \int_{\Theta^t} \theta_t u\left(c_t\left(\sigma^t\left(\theta^t\right)\right)\right) d\mu_t\left(\theta^t\right)$$

For any measurable function $\sigma^t : \Theta^t \to \Theta^t$ according to measure μ_t.

- Planning problem

$$\max \sum_{t=0}^{T} \beta^t \int_{\Theta^t} \theta_t u\left(c_t\left(\theta^t\right)\right) d\mu_t\left(\theta^t\right)$$

subject to

$$\begin{aligned}
\int_{\Theta^t} c_t\left(\theta^t\right) d\mu_t\left(\theta^t\right) &= e \\
\sum_{t=0}^{T} \beta^t \int_{\Theta^t} \theta_t u\left(c_t\left(\theta^t\right)\right) d\mu_t\left(\theta^t\right) &\geq \sum_{t=0}^{T} \beta^t \int_{\Theta^t} \theta_t u\left(c_t\left(\sigma^t\left(\theta^t\right)\right)\right) d\mu_t\left(\theta^t\right), \forall \sigma^t \\
\theta_{-1} &: \text{ given}
\end{aligned}$$

- Complicated set of incentive constraints; Need to simplify

- One-shot deviation principle: Suppose that $\beta < 1$ and u is bounded or $T < \infty$. Then an allocation is incentive compatible if and only if it satisfies one-shot incentive compatibility: $\sigma_t\left(\theta^t\right) = \theta_t$ except for a unique period $\hat{t}$ and for a positive measure of histories $\hat{\theta}^t$. With discrete processes,

$$\exists! \theta^t \sigma_t\left(\theta^t\right) \neq \theta_t$$

- with finite T, a backward induction argument works.

- With $T = \infty$, do not have to worry about deviations far away in time. same idea works. See Fudenberg and Tirole, Section 5.

- This simplifies the set of incentive constraints. How do we write them down?

Define the following utility value:

$$
\begin{aligned}
v_T\left(\theta^T\right) &= \theta_T u\left(c_T\left(\theta^T\right)\right) \\
v_t\left(\theta^t\right) &= \theta_t u\left(c_t\left(\theta^t\right)\right) + \beta \int v_{t+1}\left(\theta^t, \theta_{t+1}\right) \pi\left(d\theta_{t+1}|\theta_t\right)
\end{aligned}
$$

One-shot deviation: lie today and tell the truth from then on.

Value of telling the truth from now on:

$$
\theta_t u\left(c_t\left(\theta^t\right)\right) + \beta \int v_{t+1}\left(\theta^t, \theta_{t+1}\right) \pi\left(d\theta_{t+1}|\theta_t\right)
$$

Value of lying today and telling the truth from tomorrow on:

$$
\theta_t u\left(c_t\left(\theta^{t-1}, \hat{\theta}_t\right)\right) + \beta \int v_{t+1}\left(\theta^{t-1}, \hat{\theta}_t, \theta_{t+1}\right) \pi\left(d\theta_{t+1}|\theta_t\right)
$$

One-shot incentive compatibility:

$$
\begin{aligned}
&\theta_t u\left(c_t\left(\theta^t\right)\right) + \beta \int v_{t+1}\left(\theta^t, \theta_{t+1}\right) \pi\left(d\theta_{t+1}|\theta_t\right) \geq \\
&\qquad \theta_t u\left(c_t\left(\theta^{t-1}, \hat{\theta}_t\right)\right) + \beta \int v_{t+1}\left(\theta^{t-1}, \hat{\theta}_t, \theta_{t+1}\right) \pi\left(d\theta_{t+1}|\theta_t\right)
\end{aligned}
$$

- Planning problem

$$
w^* = \max \int v_0\left(\theta_0\right) \pi\left(d\theta_0|\theta_{-1}\right)
$$

subject to

$$
\begin{aligned}
&\theta_t u\left(c_t\left(\theta^t\right)\right) + \beta \int v_{t+1}\left(\theta^t, \theta_{t+1}\right) \pi\left(d\theta_{t+1}|\theta_t\right) = v_t\left(\theta^t\right) \\
&\int_{\Theta^t} c_t\left(\theta^t\right) d\mu_t\left(\theta^t\right) = e \\
&\theta_t u\left(c_t\left(\theta^t\right)\right) + \beta \int v_{t+1}\left(\theta^t, \theta_{t+1}\right) \pi\left(d\theta_{t+1}|\theta_t\right) \geq \\
&\qquad \theta_t u\left(c_t\left(\theta^{t-1}, \hat{\theta}_t\right)\right) + \beta \int v_{t+1}\left(\theta^{t-1}, \hat{\theta}_t, \theta_{t+1}\right) \pi\left(d\theta_{t+1}|\theta_t\right)
\end{aligned}
$$

- Would like to rewrite it recursively. For dynamic problems, very practical and useful to deal with the dual. How do we know it is equivalent. For now, let's just assume!! Most problem have no way of knowing, in this case we do!!

- Lagrange multipliers on feasibility must exist, call them λ_t.

- Then the above is equivalent to

$$\max \sum_{t=0}^{T} \lambda_t \left[e - \int_{\Theta^t} c_t\left(\theta^t\right) d\mu_t\left(\theta^t\right) \right]$$

subject to

$$\begin{aligned}
&\theta_t u\left(c_t\left(\theta^t\right)\right) + \beta \int v_{t+1}\left(\theta^t, \theta_{t+1}\right) \pi\left(d\theta_{t+1} | \theta_t\right) = v_t\left(\theta^t\right) \\
&\int v_0\left(\theta_0\right) \pi\left(d\theta_0 | \theta_{-1}\right) \geq \bar{w} \\
&\theta_t u\left(c_t\left(\theta^t\right)\right) + \beta \int v_{t+1}\left(\theta^t, \theta_{t+1}\right) \pi\left(d\theta_{t+1} | \theta_t\right) \geq \\
&\qquad \theta_t u\left(c_t\left(\theta^{t-1}, \hat{\theta}_t\right)\right) + \beta \int v_{t+1}\left(\theta^{t-1}, \hat{\theta}_t, \theta_{t+1}\right) \pi\left(d\theta_{t+1} | \theta_t\right)
\end{aligned}$$

- Notice assumption: No names; treating people fully symmetrically; what would happen if people had names?

- Suppose that $|\Theta| = N$. State variable: $\theta_{t-1}, \int v_t\left(\theta^{t-2}, \theta_{t-1}, \theta_t\right) \pi\left(d\theta_t | \hat{\theta}_{t-1}\right) = w_t\left(\theta_{t-1}, \hat{\theta}_{t-1}\right)$: the value a true type of $\hat{\theta}_{t-1}$ gets if pretends to be θ_{t-1} at $t-1$. $N+1$ state variables

$$\begin{aligned}
&P_t\left(\theta_-, w\left(\theta_-\right), \left\{w\left(\theta_-, \hat{\theta}_-\right)\right\}_{\hat{\theta}_- \neq \theta_-}\right) = \\
&\qquad \max \int_{\Theta} \left[e - c\left(\theta\right) + \frac{\lambda_{t+1}}{\lambda_t} P_{t+1}\left(\theta, w'\left(\theta\right), \left\{w'\left(\theta, \hat{\theta}\right)\right\}\right) \right] \pi\left(d\theta | \theta_-\right)
\end{aligned}$$

subject to

$$\begin{aligned}
\int \left[\theta u\left(c\left(\theta\right)\right) + \beta w'\left(\theta\right)\right] \pi\left(d\theta | \theta_-\right) &= w\left(\theta_-\right) \\
\int \left[\theta u\left(c\left(\theta\right)\right) + \beta w'\left(\theta\right)\right] \pi\left(d\theta | \hat{\theta}_-\right) &= w\left(\theta_-, \hat{\theta}_-\right) \\
\theta u\left(c\left(\theta\right)\right) + \beta w'\left(\theta\right) &\geq \theta u\left(c\left(\hat{\theta}\right)\right) + \beta w'\left(\theta, \hat{\theta}\right) \\
w\left(\theta, \hat{\theta}\right) &: \text{ feasible}
\end{aligned}$$

- Period 0 problem

$$\max_{w\left(\hat{\theta}_{-1}\right)} \int_{\Theta} \left[e - c\left(\theta\right) + \frac{\lambda_1}{\lambda_0} P_1\left(\theta, w'\left(\theta\right), \left\{w'\left(\theta, \hat{\theta}\right)\right\}\right) \right] \pi\left(d\theta | \theta_{-1}\right)$$

subject to

$$\begin{aligned}
\bar{w} &= \int \left[\theta_0 u\left(c\left(\theta_0\right)\right)+\beta w'\left(\theta_0\right)\right] \pi\left(d\theta_0 | \theta_{-1}\right) \\
w\left(\hat{\theta}_{-1}\right) &= \int \left[\theta_0 u\left(c\left(\theta_0\right)\right)+\beta w'\left(\theta_0\right)\right] \pi\left(d\theta_0 | \hat{\theta}_{-1}\right) \\
w\left(\hat{\theta}_{-1}\right) &: \text{ feasible}
\end{aligned}$$

- How do we find the set of feasible $w\left(\theta, \hat{\theta}\right)$'s; A procedure similar to Abreu, Pearce and Stachetti (To read more see Fernandes and Phelan, JET, 2000). The general problem with persistence is very complicated. Later we will see a special case(θ_t is a random walk) that we can actually solve by hand!

4.1 I.I.D. Shocks

- What happens if we have i.i.d θ_t's?

$$\begin{aligned}
w\left(\theta_{t-1}, \hat{\theta}_{t-1}\right) &= \int v_t\left(\theta^{t-2}, \theta_{t-1}, \theta_t\right) \pi\left(d\theta_t | \hat{\theta}_{t-1}\right) \\
&= \int v_t\left(\theta^{t-2}, \theta_{t-1}, \theta_t\right) \pi\left(d\theta_t | \theta_{t-1}\right) = w\left(\theta_{t-1}\right)
\end{aligned}$$

Number of state variables = 1

$$P_t(w) = \int \left[e - c(\theta) + \frac{\lambda_{t+1}}{\lambda_t} P_{t+1}\left(w'(\theta)\right)\right] d\pi(\theta)$$

subject to

$$\begin{aligned}
\int \left[\theta u\left(c\left(\theta\right)\right)+\beta w'\left(\theta\right)\right] d\pi\left(\theta\right) &= w \\
\theta u\left(c\left(\theta\right)\right)+\beta w'\left(\theta\right) &\geq \theta u\left(c\left(\hat{\theta}\right)\right)+\beta w'\left(\hat{\theta}\right)
\end{aligned}$$

Have to find λ_t's together with w_0 such that $P_0\left(w_0\right)=0$ and that $\int c_t\left(\theta^t\right) d\mu_t\left(\theta^t\right) = e$.

- How do we know that solution to the above problem is unique? IC's make the problem potentially non-covex.

- Let's do a transformation as before:

$$P_t(w) = \int \left[-C\left(u\left(\theta\right)\right) + \frac{\lambda_{t+1}}{\lambda_t} P_{t+1}\left(w'(\theta)\right)\right] d\pi(\theta)$$

subject to

$$\begin{aligned}
\int \left[\theta u(\theta) + \beta w'(\theta)\right] d\pi(\theta) &= w \\
\theta u(\theta) + \beta w'(\theta) &\geq \theta u\left(\hat{\theta}\right) + \beta w'\left(\hat{\theta}\right)
\end{aligned}$$

Constraint set is linear now and the period objective is strictly concave. The usual method from SLP can be used to show that the value function is (strictly) concave.

- What happens when $T = 0$?
- What happens when $T = 1$?
- It turns out for many practical utility functions we have closed form solutions. For now, let's assume that $q_{t+1} = \frac{\lambda_{t+1}}{\lambda_t}$ is constant over time, q, and also that $T = \infty$. It is fairly easy to generalize the following analysis to the case where q_{t+1} is changing over time and finite horizon. Given this, we can drop the subscript t:

$$P(w) = \max \int \left[-c(\theta) + qP\left(w'(\theta)\right)\right] d\pi(\theta)$$

subject to

$$\begin{aligned}
\int \left[\theta u(c(\theta)) + \beta w'(\theta)\right] d\pi(\theta) &= w \\
\theta u(c(\theta)) + \beta w'(\theta) &\geq \theta u\left(c\left(\hat{\theta}\right)\right) + \beta w'\left(\hat{\theta}\right)
\end{aligned}$$

It is helpful to work with utilities $u(\theta) = u(c(\theta))$ or $c(\theta) = C(u(\theta))$.

$$P(w) = \max \int \left[-C(u(\theta)) + qP\left(w'(\theta)\right)\right] d\pi(\theta)$$

subject to

$$\begin{aligned}
\int \left[\theta u(\theta) + \beta w'(\theta)\right] d\pi(\theta) &= w \\
\theta u(\theta) + \beta w'(\theta) &\geq \theta u\left(\hat{\theta}\right) + \beta w'\left(\hat{\theta}\right)
\end{aligned}$$

Now, we can see that the constraint set is linear of degree one in all the variables. That suggests that the policy functions should be homogenous too, as long as the objective is homothetic. We can consider three cases:

  - $u(c) = \log c$; In this case, $C(u) = e^u$, adding a constant to either $u(\theta)$ or $w'(\theta)$ would not violate the incentive constraints. This suggest that the policy functions and value functions are homogeneous too. Guess:

$$\begin{aligned} u(\theta, w) &= \hat{u}(\theta) + \frac{1-\beta}{E\theta} w \\ w'(\theta, w) &= \hat{w}(\theta) + w \\ P(w) &= -Ae^{\frac{1-\beta}{E\theta} w} \end{aligned}$$

Problem becomes

$$\max e^{\frac{1-\beta}{E\theta} w} \int \left[-e^{\hat{u}(\theta)} - qAe^{\frac{1-\beta}{E\theta} \hat{w}(\theta)} \right] d\pi(\theta)$$

subject to

$$\begin{aligned} \int [\theta \hat{u}(\theta) + \beta \hat{w}(\theta)] \, d\pi(\theta) &= 0 \\ \theta \hat{u}(\theta) + \beta \hat{w}'(\theta) &\geq \theta \hat{u}(\hat{\theta}) + \beta \hat{w}(\hat{\theta}) \end{aligned}$$

Notice that the solution to the above problem is independent of w. So the objective must take value given by $\hat{A}(A)$. By contraction mapping theorem, the function $\hat{A}(A)$ must have a unique fixed point and hence the guess is verified. We can describe the policy functionsin for consumption by

$$c(\theta, w; q) = e^{\hat{u}(\theta; q)} e^{\frac{1-\beta}{E\theta} w}$$

Note that q is an endogenous object and need to be determined in order to fully describe the solution to the original planning problem. Will get to this soon.

Notice also what the baseline planning problem looks like. It looks a lot like our original risk sharing problem with apples and bananas. This problem keeps coming up all the time!

We can also use the above policy functions repeatedly, to see

$$c_t(\theta^t, w_0; q) = e^{\hat{u}(\theta_t; q)} e^{\frac{1-\beta}{E\theta} \left[w_0 + \sum_{s=0}^{t-1} \hat{w}(\theta_s; q) \right]}$$

- $u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \sigma \neq 1$; In this case, $C(u) = ((1-\sigma)u)^{\frac{1}{1-\sigma}}$. Multiplying the constraint set by a positive factor doesn't change anything, so we can make the following guess

$$\begin{aligned} u(\theta, w) &= \hat{u}(\theta) \cdot (1-\sigma) w \\ w'(\theta, w) &= \hat{w}(\theta) \cdot (1-\sigma) w \\ P(w) &= A[(1-\sigma) w]^{\frac{1}{1-\sigma}} \end{aligned}$$

Putting the $1-\sigma$ there makes sure that the factor is positive. To see why it works, under the above guess the problem becomes

$$\max\left[(1-\sigma)w\right]^{\frac{1}{1-\sigma}}\int\left[-\left[(1-\sigma)\hat{u}(\theta)\right]^{\frac{1}{1-\sigma}}-qA\left[(1-\sigma)\hat{w}(\theta)\right]^{\frac{1}{1-\sigma}}\right]d\pi(\theta)$$

subject to

$$\begin{aligned}\int\left[\theta\hat{u}(\theta)+\beta\hat{w}(\theta)\right]d\pi(\theta) &= \frac{1}{1-\sigma}\\ \theta\hat{u}(\theta)+\beta\hat{w}(\theta) &\geq \theta\hat{u}\left(\hat{\theta}\right)+\beta\hat{w}\left(\hat{\theta}\right)\end{aligned}$$

As before, the solution is independent of w. Hence,

$$c(\theta,w;q)=\left[(1-\sigma)w\right]^{\frac{1}{1-\sigma}}\left[(1-\sigma)\hat{u}(\theta;q)\right]^{\frac{1}{1-\sigma}}$$

and

$$c_t\left(\theta^t,w_0;q\right)=\left[(1-\sigma)w_0\right]^{\frac{1}{1-\sigma}}\left[(1-\sigma)\hat{u}(\theta_t;q)\right]^{\frac{1}{1-\sigma}}\prod_{s=0}^{t-1}\left[(1-\sigma)\hat{w}(\theta_s;q)\right]^{\frac{1}{1-\sigma}}$$

- $u(c)=-\frac{1}{\psi}e^{-\psi c}$; In this case $C(u)=-\frac{1}{\psi}\log(-\psi u)$. Again multiplying the constraint set by a positive factor does not change anything, so we have the following guess

$$\begin{aligned}u(\theta,w) &= \hat{u}(\theta)\cdot(-w)\\ w'(\theta,w) &= \hat{w}(\theta)\cdot(-w)\\ P(w) &= \frac{1}{(1-q)\psi}\log(-w)+B\end{aligned}$$

for some constant B. Given the guess, the problem becomes

$$\begin{aligned}&\max\int\left[\frac{1}{\psi}\log\left[-\psi\hat{u}(\theta)\cdot(-w)\right]+\frac{1}{\psi(1-q)}q\log\left[-\psi\hat{w}(\theta)\cdot(-w)\right]+qB\right]d\pi(\theta)\\ &=\frac{1}{\psi(1-q)}\log(-w)+\int\left[\frac{1}{\psi}\log\left[-\psi\hat{u}(\theta)\right]+\frac{1}{\psi(1-q)}q\log\left[-\psi\hat{w}(\theta)\right]+qB\right]d\pi(\theta)\end{aligned}$$

subject to

$$\begin{aligned}\int\left[\theta\hat{u}(\theta)+\beta\hat{w}(\theta)\right]d\pi(\theta) &= -1\\ \theta\hat{u}(\theta)+\beta\hat{w}(\theta) &\geq \theta\hat{u}\left(\hat{\theta}\right)+\beta\hat{w}\left(\hat{\theta}\right)\end{aligned}$$

This confirms the guess. So we have

$$c(\theta, w; q) = -\frac{1}{\psi}\log\left(-\psi\hat{u}(\theta; q)\cdot(-w)\right)$$

and

$$c_t\left(\theta^t, w_0; q\right) = -\frac{1}{\psi}\log\left(-\psi\hat{u}(\theta; q)\right) - \frac{1}{\psi}\log\left(-w_0\right) - \frac{1}{\psi}\sum_{s=0}^{t-1}\log\left(-\hat{w}(\theta_s; q)\right)$$

A note about the CARA case: Notice that utility function is given by

$$-\frac{1}{\psi}\sum_{t=0}^{T}\beta^t\theta_t e^{-\psi c_t}$$

Let $\varphi_t = -\frac{1}{\psi}\log\theta_t$. Then, the above problem becomes

$$-\frac{1}{\psi}\sum_{t=0}^{T}\beta^t e^{-\psi(c_t+\varphi_t)}$$

So the taste shock problem above becomes equivalent to an income fluctuation problem with the shock process φ_t. For more on this see Thomas and Worrall(1990, JET). The income fluctuation problem in general turns out to be harder to handle than the taste shock model. It is a lot harder to show concavity; have to make assumptions about the direction that the incentive constraints bind. Thomas and Worrall are being very careful about this although the paper has some small mistakes. It is a must-read.

- To completely solve the original planning problem, we need to find q and w_0. For this, we will use feasibility:

$$\int_{\Theta^t} c_t\left(\theta^t, w_0; q\right) d\mu_t\left(\theta^t\right) = e, \forall t \geq 0$$

Now, let's what this implies for each case:

  – Log case

$$\int_{\Theta^t} c_t\left(\theta^t, w_0; q\right) d\mu_t\left(\theta^t\right) = e$$

or

$$\int_{\Theta^t}\left[e^{\hat{u}(\theta_t;q)}e^{\frac{1-\beta}{E\theta}\left[w_0+\sum_{s=0}^{t-1}\hat{w}(\theta_s;q)\right]}\right] d\mu_t\left(\theta^t\right) = e$$

Since θ_t's are independent, we can rewrite the above integral as

$$e^{\frac{1-\beta}{E\theta}w_0}\Pi_{s=0}^{t-1}\int e^{\frac{1-\beta}{E\theta}\hat{w}(\theta;q)}d\pi(\theta)\int e^{\hat{u}(\theta;q)}d\pi(\theta) = e$$

This will come down to the followig equations(why?)

$$\begin{aligned} \int e^{\frac{1-\beta}{E\theta}\hat{w}(\theta;q)} d\pi(\theta) &= 1 \\ P(w_0) + \frac{e}{1-q} &= 0 \end{aligned}$$

These two equations pin down q and w_0.

– CRRA case:

$$\int_{\Theta^t} c_t\left(\theta^t, w_0; q\right) d\mu_t\left(\theta^t\right) = e$$

or

$$\int_{\Theta^t} \left[(1-\sigma) w_0\right]^{\frac{1}{1-\sigma}} \left[(1-\sigma)\hat{u}(\theta_t;q)\right]^{\frac{1}{1-\sigma}} \prod_{s=0}^{t-1} \left[(1-\sigma)\hat{w}(\theta_s;q)\right]^{\frac{1}{1-\sigma}} d\mu_t\left(\theta^t\right) = e$$

Similarly, we have

$$\begin{aligned} \int \left[(1-\sigma)\hat{w}(\theta;q)\right]^{\frac{1}{1-\sigma}} d\pi(\theta) &= 1 \\ P(w_0) + \frac{e}{1-q} &= 0 \end{aligned}$$

– CARA case:

$$\int_{\Theta^t} c_t\left(\theta^t, w_0; q\right) d\mu_t\left(\theta^t\right) = e$$

or

$$\int_{\Theta^t} \left[-\frac{1}{\psi}\log\left(-\psi\hat{u}(\theta;q)\right) - \frac{1}{\psi}\log(-w_0) - \frac{1}{\psi}\sum_{s=0}^{t-1}\log\left(-\hat{w}(\theta_s;q)\right)\right] d\mu_t\left(\theta^t\right) = e$$

Hence,

$$\begin{aligned} \int \log\left(-\hat{w}(\theta;q)\right) d\pi(\theta) &= 0 \\ P(w_0) + \frac{e}{1-q} &= 0 \end{aligned}$$

For more general formulations, it is possible to write the above but wouldn't be able to bring them down to single equations.

Question: What happens when the problem is non-stationary, i.e, when $T < \infty$ or when q_t's are not constant.

- Now, we have fully characterized the solution to original planning problem. One way to think about the model: what is the amount of optimal inequality? How much inequality in consumption and wealth should societies tolerate given heterogeneity in taste or income? Let's see what the model implies about optimal amount of inequality in consumption:

- Let's focus on the log case:

$$c_t(\theta_t) = e^{\hat{u}(\theta_t;q)} e^{\frac{1-\beta}{E\theta}\left[w_0 + \sum_{s=0}^{t-1} \hat{w}(\theta_s;q)\right]}$$

where w_0 and q are given by the above equations. or

$$\log c_t(\theta_t) = \hat{u}(\theta_t;q) + \frac{1-\beta}{E[\theta]}\left[w_0 + \sum_{s=0}^{t-1} \hat{w}(\theta_s;q)\right]$$

$\log c_t - \log \hat{u}(\theta_t;q)$ is a random walk with drift:

$$\frac{1-\beta}{E[\theta]} E[\hat{w}(\theta_s;q)]$$

What is the sign of the drift? We know that

$$\int e^{\frac{1-\beta}{E\theta}\hat{w}(\theta;q)} d\pi(\theta) = 1$$

Take logs

$$\log \int e^{\frac{1-\beta}{E\theta}\hat{w}(\theta;q)} d\pi(\theta) = 0$$

By Jensen's inequality and since log is concave,

$$E[\log X] < \log E[X]$$

when X is non-constant almost surely. So

$$\int \log\left[e^{\frac{1-\beta}{E\theta}\hat{w}(\theta;q)}\right] d\pi(\theta) < \log \int e^{\frac{1-\beta}{E\theta}\hat{w}(\theta;q)} d\pi(\theta) = 0$$

or

$$\int \frac{1-\beta}{E\theta}\hat{w}(\theta;q)\, d\pi(\theta) < 0$$

Hence the drift is negative. So consumption is a random walk with a negative drift. So the variance of consumption converges to infinity and consumption converges to 0 almost surely (can show this for other cases too). **Immiseration**. Disappointing theory of optimal inequality!! This property is very robust in dynamic contracting models– general preferences, finite horizon(increasing variance) or even shock process (how persistent it is, we will see an example for this soon)

- Alternative theories to address this issue: can we modify the model to think about optimal inequality? (if we have time, we'll talk about them)

– Unequal discounting: Interpret each period as a generation; The planner puts more weight on future generations than households: Phelan(2006, Restud), Farhi and Werning(2007, JPE)

– Lack of Commitment: Planner cannot committ to future contracts; it realizes that inequality is really high; will reneg. Can be shown to be equivalent to a version of the above; Sleet and Yeltekin(RED, 2006)

– Fertility: Generational interpretation; planner can use both number of children and promised utility to provide incentives; will use fertility more heavily and hence there is a stationary distribution; Hosseini, Jones, and Shourideh(2010)

4.2 Persistent Shocks

- What happens with persistence? As we saw, the problem with persistence is impossible to solve because the state space becomes too large. Why? because we need to know the deviation values for households, when they lie. But when there is persistence, the household know the distribution of future payoffs but the planner does not. Easiest way to deal with this was to just stick those deviation values into the state space. But then the problem becomes impossible to solve.

- Another way of doing this is to see that truth-telling values are related to each other through the future stream of consumption so may be we can use some properties of the shock process to relate these two values to each other. It turns out that there is a special case where we can do this,the case where θ_t is a geometric random walk

$$\log \theta_{t+1} = \log \theta_t + \log \varepsilon_{t+1}$$

Now, just think about a two period example. Moreover, assume that we represent the allocations in terms of innovation shocks: ε_t, $\left\{c_0\left(\varepsilon_0\right), c_1\left(\varepsilon^1\right)\right\}$

The future utility of telling the truth, type ε_0

$$w\left(\varepsilon_0\right) = \int \theta_{-1}\varepsilon_0\varepsilon_1 u\left(c_1\left(\varepsilon_0, \varepsilon_1\right)\right) dG\left(\varepsilon_1\right)$$

The future utility of lying, ε_0 pretenfing to be $\hat{\varepsilon}_0$

$$w\left(\hat{\varepsilon}_0, \varepsilon_0\right) = \int \theta_{-1}\varepsilon_0\varepsilon_1 u\left(c_1\left(\hat{\varepsilon}_0, \varepsilon_1\right)\right) dG\left(\varepsilon_1\right)$$

The future utility of telling the truth type $\hat{\varepsilon}_0$

$$w\left(\hat{\varepsilon}_0\right) = \int \theta_{-1}\hat{\varepsilon}_0\varepsilon_1 u\left(c_1\left(\hat{\varepsilon}_0, \varepsilon_1\right)\right) dG\left(\varepsilon_1\right)$$

It is easy to see that

$$w\left(\hat{\varepsilon}_0, \varepsilon_0\right) = \frac{\varepsilon_0}{\hat{\varepsilon}_0} w\left(\hat{\varepsilon}_0\right)$$

and hence the incentive constraint is given by

$$\theta_{-1}\varepsilon_0 u\left(c_0\left(\varepsilon_0\right)\right) + \beta w\left(\varepsilon_0\right) \geq \theta_{-1}\varepsilon_0 u\left(c_0\left(\hat{\varepsilon}_0\right)\right) + \beta \frac{\varepsilon_0}{\hat{\varepsilon}_0} w\left(\hat{\varepsilon}_0\right)$$

Notice that in the above two period example, if we let persistence to be $\rho < 1$, or

$$\log \theta_{t+1} = \rho \log \theta_t + \varepsilon_{t+1}$$

then

$$w\left(\hat{\varepsilon}_0, \varepsilon_0\right) = \left(\frac{\varepsilon_0}{\hat{\varepsilon}_0}\right)^{\rho} w\left(\hat{\varepsilon}_0\right)$$

and the incentive constraint can be written as

$$\theta_{-1}^{\rho}\varepsilon_0 u\left(c_0\left(\varepsilon_0\right)\right) + \beta w\left(\varepsilon_0\right) \geq \theta_{-1}^{\rho}\varepsilon_0 u\left(c_0\left(\hat{\varepsilon}_0\right)\right) + \beta \left(\frac{\varepsilon_0}{\hat{\varepsilon}_0}\right)^{\rho} w\left(\hat{\varepsilon}_0\right) \tag{1}$$

- In case that $\rho = 1$, it is easy to extend the above analysis to a multi-period environment. The reason is that promised utility and threat keeping utility are given by

$$w\left(\varepsilon^{t-1}, \hat{\varepsilon}_t\right) = \sum_{s=t+1}^{T} \beta^{s-t-1} \int \cdots \int \theta_{-1}\varepsilon_0 \cdots \varepsilon_{t-1}\hat{\varepsilon}_t\varepsilon_{t+1} \cdots \varepsilon_s u\left(c_s\left(\varepsilon^{t-1}, \hat{\varepsilon}_t, \varepsilon_{t+1}, \cdots, \varepsilon_s\right)\right) dG\left(\varepsilon_{t+1}\right) \cdots$$

$$\hat{w}\left(\varepsilon^{t}; \hat{\varepsilon}_t\right) = \sum_{s=t+1}^{T} \beta^{s-t-1} \int \cdots \int \theta_{-1}\varepsilon_0 \cdots \varepsilon_{t-1}\varepsilon_t\varepsilon_{t+1} \cdots \varepsilon_s u\left(c_s\left(\varepsilon^{t-1}, \hat{\varepsilon}_t, \varepsilon_{t+1}, \cdots, \varepsilon_s\right)\right) dG\left(\varepsilon_{t+1}\right) \cdots$$

$$w\left(\varepsilon^{t}; \hat{\varepsilon}_t\right) = \frac{\varepsilon_t}{\hat{\varepsilon}_t} w\left(\varepsilon^{t-1}, \hat{\varepsilon}_t\right)$$

- Note that the above formulation with persistence $\rho < 1$ does not work any more. This is because

$$w\left(\varepsilon^{t-1}, \hat{\varepsilon}_t\right) = \sum_{s=t+1}^{T} \beta^{s-t-1} \int \cdots \int \theta_{-1}^{\rho^{s+1}} \varepsilon_0^{\rho^{s}} \cdots \varepsilon_{t-1}^{\rho^{s-t+1}} \hat{\varepsilon}_t^{\rho^{s-t}} \varepsilon_{t+1}^{\rho^{s-t-1}} \cdots \varepsilon_s u\left(c_s\left(\varepsilon^{t-1}, \hat{\varepsilon}_t, \varepsilon_{t+1}, \cdots, \varepsilon_s\right)\right) dG(\varepsilon$$

$$\hat{w}\left(\varepsilon^{t}; \hat{\varepsilon}_t\right) = \sum_{s=t+1}^{T} \beta^{s-t-1} \int \cdots \int \theta_{-1}^{\rho^{s+1}} \varepsilon_0^{\rho^{s}} \cdots \varepsilon_{t-1}^{\rho^{s-t+1}} \varepsilon_t^{\rho^{s-t}} \varepsilon_{t+1}^{\rho^{s-t-1}} \cdots \varepsilon_s u\left(c_s\left(\varepsilon^{t-1}, \hat{\varepsilon}_t, \varepsilon_{t+1}, \cdots, \varepsilon_s\right)\right) dG(\varepsilon$$

- Now, let's see what happens in the infinite horizon case – now only a two dimensional state variable θ_- and w.

$$P(w,\theta_-) = \max \int \left[-C(u(\theta)) + qP\left(w'(\theta),\theta\right)\right] dF(\theta|\theta_-)$$

subject to

$$\begin{aligned} \int \left[\theta u(\theta) + \beta w'(\theta)\right] dF(\theta|\theta_-) &= w \\ \theta u(\theta) + \beta w'(\theta) &\geq \theta u(\hat{\theta}) + \beta \frac{\theta}{\hat{\theta}} w'(\hat{\theta}) \end{aligned}$$

- Further characterization: conjecture $P(w,\theta_-) = \hat{P}\left(\frac{w}{\theta_-}\right)$. Why?

  Let $v = \frac{w}{\theta_-}$. We can rewrite the problem as

$$\max \int \left[-C(u(\theta)) + q\hat{P}\left(v'(\theta)\right)\right] \frac{1}{\theta_-} g\left(\frac{\theta}{\theta_-}\right) d\theta$$

subject to

$$\begin{aligned} \int \left[\theta u(\theta) + \beta\theta v'(\theta)\right] \frac{1}{\theta_-} g\left(\frac{\theta}{\theta_-}\right) d\theta &= \theta_- v \\ \theta u(\theta) + \beta\theta v'(\theta) &\geq \theta u(\hat{\theta}) + \beta\theta v'(\hat{\theta}) \end{aligned}$$

or

$$\max \int \left[-C(\hat{u}(\varepsilon)) + q\hat{P}\left(\hat{v}'(\varepsilon)\right)\right] g(\varepsilon) d\varepsilon$$

subject to

$$\begin{aligned} \int \theta_- \left[\varepsilon\hat{u}(\varepsilon) + \beta\varepsilon\hat{v}'(\varepsilon)\right] g(\varepsilon) d\varepsilon &= \theta_- v \\ \hat{u}(\varepsilon) + \beta\hat{v}'(\varepsilon) &\geq \hat{u}(\varepsilon) + \beta\hat{v}'(\varepsilon) \end{aligned}$$

Notice that the incentive constraint implies that $\hat{u}(\varepsilon) + \beta\hat{v}'(\varepsilon)$ is constant. So we can combine promise promise keeping and IC's and rewrite them as

$$\hat{u}(\varepsilon) + \beta\hat{v}'(\varepsilon) = \frac{v}{E[\varepsilon]}$$

Now, look at the above problem, This problem is independent of θ_-. So by contraction mapping theorem, the solution to the above function equation only depends on v. So we can rewrite the recursive problem by

$$\hat{P}(v) = \max \int \left[-C(\hat{u}(\varepsilon)) + q\hat{P}\left(\hat{v}'(\varepsilon)\right)\right] g(\varepsilon) d\varepsilon$$

subject to

$$\hat{u}(\varepsilon) + \beta \hat{v}'(\varepsilon) = \frac{v}{E[\varepsilon]}$$

It is clear that in the solution to the above problem

$$\begin{aligned} \hat{u}(\varepsilon) &= \hat{u}(\varepsilon') \\ \hat{v}'(\varepsilon) &= \hat{v}'(\varepsilon') \end{aligned}$$

No INSURANCE.

If the steady state of the above problem is given by v^* and q is such that $\hat{u}(\varepsilon, v^*) = e$, then

$$\begin{aligned} c_t(\theta^t) &= e \\ w_t(\theta^t) &= \theta_t v^* \end{aligned}$$

- Why? loose intuition: in this problem the planner uses differences in marginal rate of substitution to provide incentives. With geometric random walk, the marginal rate of substituition is the same across all types. No incentive for risk sharing can be provided.

- This can be more clear by looking at the incentive constraint (1). Suppose that $\theta_{-1} = 1$ and let $v = \varepsilon_0^{-\rho} w(\varepsilon_0)$. Given this the incentive constraint becomes

$$\varepsilon_0 u_0(\varepsilon_0) + \beta \varepsilon_0^{\rho} v(\varepsilon_0) \geq \varepsilon_0 u_0(\hat{\varepsilon}_0) + \beta \varepsilon_0^{\rho} v(\hat{\varepsilon}_0)$$

  Now think about the trade-off between current utility, u_0, and future 'utility', $v(\varepsilon_0)$. The slope of the indifference curve for a person of type ε_0 is given by

$$-\beta \varepsilon_0^{\rho - 1}$$

  When $\rho = 1$, the slope of these indifference curves are all equal. Hence the planner cannot exploit these differences to provide incentives. So the planner must treat all agents similarly. The lower ρ, the higher it is the difference between these slopes. Hence, for lower ρ's, it is easier to provide incentives for truth telling and easier to separate types.

- In the income shock interpretation of this model – with CARA, constant consumption means autarky. So the variance of consumption still goes off to infinity in the long run – geometric random walk. See Williams(2011, Econometrica) for a continuous time version of this; Hopefully we will discuss continuous time soon.

5 Application: Optimal Unemployment Insurance

- We have learned the basic way to deal with dynamic contracting/mechanism design problems. The first application of those methods, we will consider is optimal unemployment insurance.
- The main idea is that unemplyment insurance is desirable – it ensures worker against income shocks, but it provides perverse incentives for people to look for jobs.
- Questions: How should we design optimal unemployment insurance policy to balance insurance and incentives? How should benefits depend on the unemployment spell? What should be the replacement ratio? etc.
- How do we model this tention? job seeking effort: private information with effort being costly
- If provide full insurance against unemployment, nobody would ever look for a job.
- Can think of this as a dynamic contracting problem – Based on Hopenhayn and Nicolini (1997, JPE).
- Basic environment: one agent and one unemployment insurance agency.(Can also think of a continuum; doesn't really make a difference in the way we write down the contracting problem)
- Agent's preference:

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) - a_t \right]$$

- Probability of finding a job: $p\left(a_t\right)$:concave
- Job: a permanent wage w.(no firing); Employment: an absorbing state
- History $h_t = \left(s_0, \cdots, s_{t-1}\right)$ where s_j is 0 if unemployed at j and 1 otherwise.
- Contracts: $c_t\left(h_t\right)$, $a_t\left(h_t\right)$. $\pi_t\left(h_t; a_t\right)$ distribution of histories at t
- Unemployment agency's problem wants to maximize household utility

$$\max \sum_{t=0}^{\infty} \beta^t \int \left[u\left(c_t\left(h_t\right)\right) - \left(1 - s_t\left(h_t\right)\right) a_t\left(h_t\right) \right] d\pi\left(h_t; a_t\right)$$

- Total resources available to the insurance agency C:

$$\sum_{t=0}^{\infty} \beta^t \int \left[c_t\left(h_t\right) - w s_t\left(h_t\right) \right] d\pi\left(h_t; a_t\right) = C$$

- Full information
$$c_t(h_t) = c$$
consumption is constant. a^* solves
$$a^* = \arg\max_a \sum_{t=0}^{\infty} \beta^t (1-p(a))^t \left[\frac{\beta u'(c) w}{1-\beta} - a\right]$$

- Private Information: revelation principle still applies; the planner suggest an action to the agent; that action has to be optimal, i.e.,
$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t \int [u(c_t(h_t)) - (1-s_t(h_t)) a_t(h_t)] \, d\pi(h_t; a_t) \geq \\ \sum_{t=0}^{\infty} \beta^t \int [u(c_t(h_t)) - (1-s_t(h_t)) \hat{a}_t(h_t)] \, d\pi(h_t; \hat{a}_t)\end{aligned}$$
for any other action profile $\hat{a}_t(h_t)$.

- As before, we can focus on one-shot deviations. They only matter after histories for which the worker has been unemployed. We can also define promise utility
$$v^e(h_t), v^u(h_t)$$
where $v^e(h_t)$ is the promised utility of a person who finds a job at the begining of $t+1$; $v^u(h_t)$ is the promised utility for a person who does not find a job; one-shot IC is given by
$$\begin{aligned}&-a_t(h_t) + \beta [p(a_t(h_t)) v^e(h_t) + (1-p(a_t(h_t))) v^u(h_t)] \geq \\ &\quad -\hat{a} + \beta [p(\hat{a}) v^e(h_t) + (1-p(\hat{a})) v^u(h_t)]\end{aligned}$$

- We are also gonna look at the dual as usual; so the unemployment insurance agency minimizes the cost of insuring the agent subject to providing a least level of utility.

- We are gonna jump right into the recursive formulation: Suppose $P^u(v)$ and $P^e(v)$ are two value functions associated with employment and unemployment, then we have
$$P^u(v) = \min c + \beta [p(a) P^e(v^e) + (1-p(a)) P^u(v^u)]$$
subject to
$$\begin{aligned} u(c) - a + \beta [p(a) v^e + (1-p(a)) v^u] &= v \\ -a + \beta [p(a) v^e + (1-p(a)) v^u] &\geq -\hat{a} + \beta [p(\hat{a}) v^e + (1-p(\hat{a})) v^u] \end{aligned}$$

- As long as $v^e > v^u$, $-\hat{a} + \beta \left[p\left(\hat{a}\right) v^e + \left(1 - p\left(\hat{a}\right)\right) v^u \right]$ is a concave function. We can replace by its first order condition

$$1 = \beta p'\left(a\right)\left(v^e - v^u\right)$$

- $$P^u\left(v\right) = \min c + \beta \left[p\left(a\right) P^e\left(v^e\right) + \left(1 - p\left(a\right)\right) P^u\left(v^u\right) \right]$$

  subject to

$$\begin{aligned} u\left(c\right) - a + \beta\left[p\left(a\right) v^e + \left(1 - p\left(a\right)\right) v^u\right] &= v \\ \beta p'\left(a\right)\left(v^e - v^u\right) &= 1 \end{aligned}$$

- Solving for $P^e\left(v\right)$ is really easy; when people get a job their consumption is constant. Also they do not have to look for jobs:

$$P^e\left(v\right) = \min c - w + \beta P^e\left(v'\right)$$

  subject to

$$u\left(c\right) + \beta v' = v$$

  By Euler equation c has to be constant. Hence

$$\frac{u\left(c\right)}{1 - \beta} = v$$

  and therefore

$$P^e\left(v\right) = \frac{-w + u^{-1}\left(\left(1 - \beta\right) v\right)}{1 - \beta}$$

- Now, let's turn to the unemployed state's problem. In general proving convexity of the value function is almost impossible in dynamic contracting problems. This is because of the non-convexity that incentive constraints introduce. So what Hopenhayn and Nicolini do is to assume that the value function is convex and differnetiable. Hence, we will be able to take first order conditions. They are given by – let's call consumption c^u:

$$\begin{aligned} 1 - \lambda u'\left(c^u\right) &= 0 \\ \beta p\left(a\right) P^{e\prime}\left(v^e\right) - \lambda \beta p\left(a\right) - \eta \beta p'\left(a\right) &= 0 \\ \beta\left(1 - p\left(a\right)\right) P^{e\prime}\left(v^u\right) - \lambda \beta\left[1 - p\left(a\right)\right] + \eta \beta p'\left(a\right) &= 0 \\ \beta p'\left(a\right)\left[P^e\left(v^e\right) - P^u\left(v^u\right)\right] - \lambda\left[-1 + \beta p'\left(a\right)\left[v^e - v^u\right]\right] - \eta \beta p''\left(a\right)\left[v^e - v^u\right] &= 0 \end{aligned}$$

  and an envelope condition

$$P^{u\prime}\left(v\right) = \frac{1}{u'\left(c^u\right)}$$

We can rewrite the above

$$\begin{aligned} P^{e\prime}(v^e) &= \frac{1}{u'(c^u)} + \eta \frac{p'(a)}{p(a)} \\ P^{u\prime}(v^u) &= \frac{1}{u'(c^u)} - \eta \frac{p'(a)}{1-p(a)} \end{aligned}$$

$$p'(a)\left[P^e(v^e) - P^u(v^u)\right] = \eta p''(a)\left[v^e - v^u\right]$$

- **Inverse Euler Equation.**

  Combingin the above with the envelope condition, we have the following

$$P^{u\prime}(v) = p(a) P^{e\prime}(v^e) + (1-p(a)) P^{u\prime}(v^u)$$

We can think about the above equation as an Euler Equation for the UI agency. Consider an increase in v by ε. As a result, the UI agency can increase c^u by ε or can increase v^e and v^u by $\varepsilon\beta^{-1}$. Note that the latter, does not affect the incentive constraint. The beneft of the former change to the insurance agency is given by

$$\frac{1}{u'(c^u)}\varepsilon = P^{u\prime}(v)\varepsilon$$

while the benefit of the latter perturbation is given by

$$\beta\left[p(a) P^{e\prime}(v^e) + (1-p(a)) P^{u\prime}(v^u)\right]\beta^{-1}\varepsilon$$

At the optimal allocation, the UI agency should be indifferent between these two perturbations, so we get the original equation. Notice that if we use Envelop conditions, the above equation implies that

$$\frac{1}{u'(c_t)} = E_t \frac{1}{u'(c_{t+1})}$$

This is the so-called Inverse Euler Equation. It turns out that it is a very general property of consumption in dynamic contracting problems, as long as the source of private information, here cost of effort, is separable from consumption – think about the above perturbation; what part of the argument fails if utility function was of the form $u(c,a)$?

It also has a strong implication, that these agents should be saving constrained when they are in the unemployed state. Why? This is because of Jensen's inequality

$$\frac{1}{u'(c_t)} = E_t \frac{1}{u'(c_{t+1})} \geq \frac{1}{E_t u'(c_{t+1})}$$

and hence

$$u'(c_t) \leq E_t u'(c_{t+1})$$

Note that the above inequality is strict in the unemployment state since future consumption is not degenerate. So if the agent had access to saving technology with return β^{-1}, he would like to save in order to decrease future promsied utility. The reason is that in this model, saving tightens future incentive constraints. An unemployed agent would rather save a little bit and put in less effort, lower than what the UI agency would like him/her to do. This is because of concavity of utlity function. If the agent increases his saving, his utility in the unemployed state increases more than his utility in the employed state since utility function is concave and therefore he has a lower incentive to exert effort.

- **Decreasing Benefits.**

  Consider a decentralization in which UI agency doesn't let the agents save. In this decntralization, unemployment benefit is simply $c^u(v)$ when the state is unemployed is at state v. Moreover, the agent pays unemployment taxes once he gets a job. The tax level is given by $w - c^e(v)$ at state v. Given the strucutre of the model we can define three objects:

  1. v_t^u: the promised utility for an agent who has been unemployed since $t = 0$ and is unemployed at the beginning of t.
  2. v_{t+1}^e: the promised utility for an agent who has been unemployed since $t = 0$ and finds a job at the beginning of $t + 1$.
  3. c_t^u: the consumption of a person who is unemployed at t.

  Now, whether benefits are decreasing or increasing depends on whether c_t^u is a decreasing or increasing sequence. Now consider the above FOCs. We have

$$P^{u\prime}\left(v_{t+1}^u\right) = \frac{1}{u'\left(c_t^u\right)} - \eta \frac{p'(a)}{1 - p(a)} = \frac{1}{u'\left(c_{t+1}^u\right)}$$

  So whether c_t^u is bigger or smaller than c_{t+1}^u is determined by the sign of η.

  Lemma 2. *$\eta > 0$.*

  Proof.

  Suppose not. Then

$$P^{u\prime}(v^u) \geq P^{u\prime}(v) \geq P^{e\prime}(v^e)$$

  together with convexity of the value function and local incentive compatibility implies that

$$v \leq v^u < v^e$$

Note that $c^u \geq c^e = u^{-1}\left(\left(1-\beta\right)v^e\right)$ since $\eta \leq 0$. Now, we can write

$$v - v^e = \left[u\left(c^u\right) - u\left(c^e\right)\right] - a + \beta\left[1 - p\left(a\right)\right]\left(v^u - v^e\right)$$

or

$$\begin{aligned} v - v^e + \beta\left(v^e - v^u\right) &= \left[u\left(c^u\right) - u\left(c^e\right)\right] - a + \beta p\left(a\right)\left(v^e - v^u\right) \\ v - v^u + \left(\beta - 1\right)\left(v^e - v^u\right) &= \left[u\left(c^u\right) - u\left(c^e\right)\right] - a + \beta p\left(a\right)\left(v^e - v^u\right) \end{aligned}$$

Note that the RHS of the above is positive since $c^u \geq c^e$ and that a is incentive compatible. However, the left hand side is negative since $v \leq v^u < v^e$ and $\beta < 1$. A contradiction. Q.E.D.

Since $\eta > 0$, then

$$\frac{1}{u'\left(c_t^u\right)} - \eta\frac{p'\left(a\right)}{1 - p\left(a\right)} = \frac{1}{u'\left(c_{t+1}^u\right)}$$

implies that

$$\frac{1}{u'\left(c_t^u\right)} > \frac{1}{u'\left(c_{t+1}^u\right)} \Rightarrow c_t^u > c_{t+1}^u$$

Moreover, by assumption, P^u is convex and therefore

$$v_t^u > v_{t+1}^u$$

- **Dependence of Taxes on Duration**

  The unemployment tax is $w - c^e\left(v_t^e\right)$ for an agent that found a job at t. It is very natural that v_t^e changes with t – taxes are history dependent; they depend on duration. How do they depend on the duration?

  Proposition 3. *Suppose either of the following conditions hold: i.* $-\frac{p''(a)p(a)(1-p(a))}{\beta p'(a)^3}$ *is increasing in a, ii.* $-\frac{p''(a)}{p'(a)^2}$ *is increasing in a, then taxes paid by workers increase with unemployment duration, i.e.,* $v_t^e > v_{t+1}^e$.

  Proof.

  We have

$$p'\left(a\right)\left[P^e\left(v^e\right) - P^u\left(v^u\right)\right] = \eta p''\left(a\right)\frac{1}{\beta p'\left(a\right)}$$

  and

$$\eta = \left[P^{e\prime}\left(v^e\right) - P^{u\prime}\left(v^u\right)\right]\frac{p\left(a\right)\left(1 - p\left(a\right)\right)}{p'\left(a\right)}$$

  hence

$$P^e\left(v^e\right) - P^u\left(v^u\right) = \left[P^{e\prime}\left(v^e\right) - P^{u\prime}\left(v^u\right)\right]\frac{p''\left(a\right)p\left(a\right)\left(1 - p\left(a\right)\right)}{\beta p'\left(a\right)^3}$$

Now suppose to the contrary that $v^e_{t+1} > v^e_t$, then since $v^u_{t+1} < v^u_t$. This together with IC and concavity of $p(a)$ implies that $a_{t+1} > a_t$. Under assumption *i.*,

$$-\frac{p''(a_{t+1})\, p(a_{t+1})\,(1 - p(a_{t+1}))}{\beta p'(a_{t+1})^3} > -\frac{p''(a_t)\, p(a_t)\,(1 - p(a_t))}{\beta p'(a_t)^3}$$

Moreover,

$$P^{e\prime}\left(v^e_{t+1}\right) > P^{e\prime}\left(v^e_t\right) > P^{u\prime}\left(v^e_t\right) \geq P^{u\prime}\left(v^u_{t+1}\right)$$

due to convexity of P^e and P^u. Therefore,

$$-\frac{p''(a_{t+1})\, p(a_{t+1})\,(1 - p(a_{t+1}))}{\beta p'(a_{t+1})^3}\left[P^{e\prime}\left(v^e_{t+1}\right) - P^{u\prime}\left(v^u_{t+1}\right)\right] > -\frac{p''(a_t)\, p(a_t)\,(1 - p(a_t))}{\beta p'(a_t)^3}\left[P^{e\prime}\left(v^e_t\right) -\right.$$

or

$$P^e\left(v^e_{t+1}\right) - P^u\left(v^u_{t+1}\right) < P^e\left(v^e_t\right) - P^u\left(v^u_t\right)$$

However, this is in contradiction with the fact that P^e and P^u are both increasing.

When assumption *ii.* holds, the argument is similar.

- Intuitively, absent incentive constraints and due to insurance reasons, the planner would like to equate v^u_t and v^e_t. So a decrease in v^u_t should be accompanied by a decrease in v^e_t. However, the planner might want to make sure that agents exert the efficient level of effort. So provided that the effect of keeping v^e and v^u close is not so strong on incentives, then it is efficient to have v^u_t be decreasing over time.

5.1 The Role of Saving

- Above, we assumed that the UI agency controls agents consumption. Furthermore, we showed that the agents were in general saving constrained, i.e., they would like to save on their own if they have the opportunity to do so. A natural question is what happens when we let them do so? Here we try to formulate this.

- We make the following change in the environment. Assume that the timing of the events in each period is as follows: upon entering date t the households consume, c_t, they then look for jobs by exerting effert, and will realize whether they find a job or not, with probability p_t. We assume that the utility function in each period is given by

$$u\left(c_t, (1 - s_{t-1})\, p_{t-1}\right)$$

Notice that we have assumed that utility is a direct function of job finding rate as opposed to effort.

- Suppose that in addition to effort, saving by agents is also hidden. A contract in this case is consisted of

$$\{c_t(h_t), \tau_t(h_t), p_t(h_t), b_t(h_t)\}$$

where $\tau_t(h_t)$ is the transfer to the agent, b_t is a suggested level of saving by the planner with $b_0 = 0$ and c_t is defined by the following budget constraint

$$c_t(h_t) + b_t(h_t) = \tau_t(h_t) + ws_t(h_t) + \beta^{-1} b_{t-1}(h_{t-1})$$

- In this case, the allocation is said to be incentive compatible if it is the solution to the following maximization problem

$$\max_{\hat{c}_t(h_t), \hat{p}_t(h_t), \hat{b}_t(h_t)} \sum_{t=0}^{\infty} \beta^t \sum_{h_t} \pi\left(h_t; \hat{p}^{t-1}\right) \left[u\left(\hat{c}_t(h_t), (1 - s_{t-1})\hat{p}_{t-1}(h_t)\right)\right]$$

subject to

$$\hat{c}_t(h_t) + \hat{b}_t(h_t) = \tau_t(h_t) + ws_t(h_t) + \beta^{-1}\hat{b}_{t-1}(h_{t-1})$$

together with an appropriate borrowing limit that is low enough so that it is never binding.

- So the UI agency now would like to minimize the cost of providing an allocation subject to incentive compatibility and an appropriate promise keeping. The UI agency also has access to the same borrowing and lending rate β^{-1}.
- First observation: without loss of generality, we can assume that in any incentive compatible allocation, $b_t(h_t) = 0$. Intuitively, as long as agents don't want to deviate, the UI agency can save for them. More technically, one can start from an allocation with $b_t(h_t) \neq 0$ and define transfers as

$$\tilde{\tau}(h_t) = \tau(h_t) + \beta^{-1} b_{t-1}(h_{t-1}) - b_t(h_t)$$

It can be shown that the contract $\{c(h_t), \tilde{\tau}(h_t), p_t(h_t), \mathbf{0}\}$ is incentive compatible.

- Incentive constraint is very complicated now. One way to simplify, as always, is to focus on the first order conditions. What are they? The first order condition with respect to a_t is given by

$$\begin{aligned} u\left(c_t^e(h_t), p_{t-1}(h_{t-1})\right) + \beta v_t^e(h_t) - u\left(c_t^u(h_t), p_{t-1}(h_{t-1})\right) - \beta v_t^u(h_t) \\ + p u_p\left(c_t^e(h_t), p_{t-1}(h_t)\right) + (1-p) u_p\left(c_t^e(h_t), p_{t-1}(h_{t-1})\right) = 0 \end{aligned}$$

when $h_t = (\underbrace{0, \cdots, 0}_{t+1 \text{ times}})$,similar to before. Additionally, there is first order condition with respect to b_t:

$$u_c\left(c_t(h_t), p_{t-1}(h_{t-1})\right) = E\left[u_c\left(c_{t+1}, (1-s_t)p_t\right) | h_t\right]$$

An Euler equation, c.f., Inverse Euler Equation.

In order for this relaxed version of incentive compatibility to be equivalent to the original definition, the objective in the original must be concave. Nothing guarantees that this is the case. In fact, here I show you a two period example where the two constraints are not equivalent – for more detailed discussion see Kocherlakota(2004, Review of Economic Dynamics).

- Consider the following two period environment: preferences are given by

$$u(c_1) + u(c_2) - v(p_1)$$

where p_1 determines the distribution of income at $t = 2$:

$$y = \begin{cases} E & \text{with probability } p \\ U & \text{with probability } 1-p \end{cases}$$

Suppose the optimal value of p is given. Then the partially planning problem is given by

$$\min_{c_1, c_E, c_U} c_1 + pc_E + (1-p)\, c_U$$

subject to

$$\begin{aligned} & u(c_1) + pu(c_E) + (1-p)\, u(c_U) - v(p) \geq w \\ & (0, p) \in \arg\max_{s, \hat{p}} u(c_1 - s) + \hat{p} u(c_E + s) + (1-\hat{p})\, u(c_U + s) - v(\hat{p}) \end{aligned} \tag{2}$$

The ralxed version of the problem is given by

$$\min_{c_1, c_E, c_U} c_1 + pc_E + (1-p)\, c_U$$

subject to

$$\begin{aligned} & u(c_1) + pu(c_E) + (1-p)\, u(c_U) - v(p) \geq w \\ & u'(c_1) = pu'(c_E) + (1-p)\, u'(c_U) \\ & v'(p) = u(c_E) - u(c_U) \end{aligned}$$

The three constraints pin down the level of consumption. For the allocation to be incentive compatible, this has to be the solution to the maximization problem (2). For this to happen, the Hessian matrix associated with that problem should be negative semi-definite at that point. This hessian matrix is given by

$$\begin{bmatrix} -\frac{1}{c_1^2} - \frac{p}{c_E^2} - \frac{1-p}{c_U^2} & \frac{1}{c_E} - \frac{1}{c_U} \\ \frac{1}{c_E} - \frac{1}{c_U} & -v''(p) \end{bmatrix}$$

When $v''(p)$ is small – very little curvature in the cost function, it is possible that the determinant of this matrix is negative so there are potential second order gains from deviations.

- Next we consider some cases where we have closed form solutions and we can check whether the first order approach is valid.
- Let's switch to the original problem. How can we rewrite the problem recursively to incorporate the Euler equation? Werning(2000) shows that if we keep track of $u_c = \lambda$ in addition to v, then we can rewrite the problem recursively. The only thing we have to worry about is that not all pairs of (v, λ) are attainable. In order to ensure this, he proposes a procedure similar to Abreu, Pearce, and Stachetti with set operators and the fixed point of that operator leads to two sets, Δ^u and Δ^e such that
$$\Delta^s = \{(v, \lambda) \,|\, (v, \lambda) \text{ is attainable by some contract}\}$$
I will side step this issue here since we will work with cases in which we can characterize these sets analytically.
- The recursive representation of the problem is given by:

  When employed at the end of the last period
$$P^e(v, \lambda) = \min_{c, v', \lambda'} c - w + \beta P^e(v', \lambda')$$
subject to
$$\begin{aligned}
&u(c, 0) + \beta v' \geq v \\
&\lambda = u_c(c, 0) \\
&(v', \lambda') \in \Delta^e
\end{aligned}$$
When unemployed at the end of the last period:
$$P^u(v, \lambda) = \min_{p, c^e, c^u, \lambda^e, v^e, \lambda^u, v^u} p\left[c^e - w + \beta P^e(v^e, \lambda^e)\right] + (1-p)\left[c^u + \beta P^u(v^u, \lambda^u)\right]$$
subject to
$$\begin{aligned}
&p\left[u(c^e, p) + \beta v^e\right] + (1-p)\left[u(c^u, p) + \beta v^u\right] \geq v \\
&p u_c(c^e, p) + (1-p) u_c(c^u, p) = \lambda \\
&\{u(c^e, p) + \beta v^e - u(c^u, p) - \beta v^u\} \\
&\quad p u_p(c^e, p) + (1-p) u_p(c^u, p) = 0 \\
&(v^s, \lambda^s) \in \Delta^s, \quad s = e, u
\end{aligned}$$

- Let's assume that the utility function satisfies the following

$$u(c,p) = -\frac{1}{\psi} e^{-\psi(c-\alpha(p))}$$

where $v(p)$ is a convex function.

- The first step is to characterize Δ^s. Given the above utility function, we have

$$u_c(c,p) = -\psi u(c,p)$$

Now, from the Euler equation

$$u_{ct} = E_t u_{c,t+1} = E_t u_{c,s}$$

and therefore

$$u_t = E_t u_{t+1} = E_t u_s, s > t$$

Therefore

$$\begin{aligned} v &= E_{t-1} \sum_{s=t}^{\infty} \beta^s u_s = \frac{u_{t-1}}{1-\beta} \\ &= -\frac{1}{\psi} \frac{u_{c,t-1}}{1-\beta} = -\frac{1}{\psi(1-\beta)} \lambda \end{aligned}$$

When we assume that households can only save the above equalities become and inequality. In the optimal contract, however, one can show that these inequalities bind, so we ignore them here.

The above equality not only characterizes the sets Δ^s, it also means that we can drop λ as state variable.

- So the recursive problem becomes:

$$P^e(v) = \min c - w + \beta P^e(v')$$

subject to

$$\begin{aligned} u(c,0) + \beta v' &= v \\ u_c(c,0) &= -\psi(1-\beta) v \end{aligned}$$

$$P^u(v) = \min p \left[c^e - w + \beta P^e(v^e)\right] + (1-p)\left[c^u + \beta P^u(v^u)\right]$$

subject to

$$\begin{aligned}
&p\left[u\left(c^{e}, p\right)+\beta v^{e}\right]+(1-p)\left[u\left(c^{u}, p\right)+\beta v^{u}\right]=v \\
&p u_{c}\left(c^{e}, p\right)+(1-p) u_{c}\left(c^{u}, p\right)=-\psi(1-\beta) v \\
&\left\{u\left(c^{e}, p\right)+\beta v^{e}-u\left(c^{u}, p\right)-\beta v^{u}\right\} \\
&\quad p u_{p}\left(c^{e}, p\right)+(1-p) u_{p}\left(c^{u}, p\right)=0
\end{aligned}$$

Note that

$$\begin{aligned}
u_{p}(c, p) &= -\psi \alpha^{\prime}(p) e^{-\psi[c-\alpha(p)]}=\psi \alpha^{\prime}(p) u(c, p) \\
u_{c}(c, p) &= \psi e^{-\psi[c-\alpha(p)]}=-\psi u(c, p)
\end{aligned}$$

So we can rewrite the above problem as

$$P^{u}(v)=\min p\left[c^{e}-w+\beta P^{e}\left(v^{e}\right)\right]+(1-p)\left[c^{u}+\beta P^{u}\left(v^{u}\right)\right] \tag{3}$$

subject to

$$\begin{aligned}
&p\left[u\left(c^{e}, p\right)+\beta v^{e}\right]+(1-p)\left[u\left(c^{u}, p\right)+\beta v^{u}\right]=v \\
&p u\left(c^{e}, p\right)+(1-p) u\left(c^{u}, p\right)=(1-\beta) v \\
&\left\{u\left(c^{e}, p\right)+\beta v^{e}-u\left(c^{u}, p\right)-\beta v^{u}\right\} \\
&\quad+\psi \alpha^{\prime}(p)\left[p u\left(c^{e}, p\right)+(1-p) u\left(c^{u}, p\right)\right]=0
\end{aligned}$$

Similar to the taste shock cases studied before, the constraint set is linear in terms of $u(c, p)$'s and v's. So we conjecture the following

$$\begin{aligned}
c^{e}(v) &= \hat{c}^{e}-\frac{1}{\psi} \log (-v) \\
c^{u}(v) &= \hat{c}^{u}-\frac{1}{\psi} \log (-v) \\
v^{e}(v) &= \hat{v}^{e} \cdot(-v) \\
v^{u}(v) &= \hat{v}^{u} \cdot(-v) \\
p(v) &= \hat{p}
\end{aligned}$$

and

$$\begin{aligned}
P^{u}(v) &= A^{u}-\frac{1}{\psi(1-\beta)} \log (-v) \\
P^{e}(v) &= A^{e}-\frac{1}{\psi(1-\beta)} \log (-v)
\end{aligned}$$

Not gonna go through the algebra. Can check again that $v^{u}(v)<v$.

- Are there simple policies that can implement the above optimal allocation? Let's try a simple policy: constant unemployment benefit $\hat{b}$, constant tax $\hat{\tau}$. Now think about a household's problem

$$V^u(a) = \max_p p\left[u\left(c^e, p\right) + \beta V^e\left(a^e\right)\right] + (1-p)\left[u\left(c^u, p\right) + \beta V^u\left(a^u\right)\right] \tag{4}$$

subject to

$$\begin{aligned} c^e + a^e &\leq w - \hat{\tau} + \beta^{-1} a \\ c^u + a^u &\leq \hat{b} + \beta^{-1} a \end{aligned}$$

together with

$$V^e(a) = \max u(c, 0) + \beta V^e\left(a'\right)$$

subject to

$$c + a' \leq w - \hat{\tau} + \beta^{-1} a$$

- We can show that there exists $\hat{\tau}$ and $\hat{b}$ so that the solution to the above problem coincides with the optimal allocations. To do so note that the functional forms for the above value functions is given by

$$\begin{aligned} V^u(a) &= -\hat{A}^u e^{-\psi\left(\beta^{-1}-1\right)a} \\ V^e(a) &= -\hat{A}^e e^{-\psi\left(\beta^{-1}-1\right)a} \end{aligned}$$

Compare this with

$$\begin{aligned} P^u(v) &= A^u - \frac{1}{\psi(1-\beta)} \log(-v) \\ P^e(v) &= A^e - \frac{1}{\psi(1-\beta)} \log(-v) \end{aligned}$$

or

$$\begin{aligned} v &= -e^{\psi(1-\beta)A^u} e^{-\psi(1-\beta)P^u(v)} \\ v &= -e^{\psi(1-\beta)A^e} e^{-\psi(1-\beta)P^e(v)} \end{aligned}$$

Now, if we set $\hat{\tau}$ and $\hat{b}$ so that $e^{\psi(1-\beta)A^u} = \hat{A}^u$ and $e^{\psi(1-\beta)A^e} = \hat{A}^e$, then $P^u(v)$ and $P^e(v)$ are associated with $\beta^{-1}a$. Moreover, it can be checked that the solution for p in (4) is identical to the solution in (3). Moreover, since the margin between c and v' in (3) is undistorted – the same as in (4), one can show that the solution to the planner's problem coincides with the solution to the agent's problem given v_0, a_0 is chosen so that $P^u(v_0) = \beta^{-1}a_0$ – we assume that the ange starts unemployed.

- To illustrate how the model changes with search explicitly modeled, we setup the model with McCall search – Shimer and Werning (2008, AER): agents draw a wage from a distribution $F(w)$; decide whether to accept or reject the offer; The value of the offer is private information. The model is very similar to the above.

- A contract is a sequence of $\{\bar{w}_t, b_t, \tau_t\}$ where b_t is the transfer to the unemployed who's been unemployed for t periods, τ_t is the tax paid by the unemployed who find a job at t and $\bar{w}_t$ is a suggestion of a reservation wage to the unemployed at date t.
- Consider an agent who comes into period t unemployed and draws wage, if he accepts the job, his utility will be

$$\frac{u(w-\tau_t)}{1-\beta}$$

and if he doesn't, his utility will be

$$u(b_t) + \beta v_{t+1}$$

where v_{t+1} is the ex-ante utility of an unemployed agent at $t+1$. For the agent to follow the recommended reservation wage, we must have

$$\frac{u(\bar{w}_t-\tau_t)}{1-\beta} = u(b_t) + \beta v_{t+1}$$

where

$$v_t = (1-F(\bar{w}_t))(u(b_t)+\beta v_{t+1}) + \int_{\bar{w}_t}^{\infty} \frac{u(w-\tau_t)}{1-\beta} dF(w)$$

6 Application: Optimal Dynamic Taxation

- In this part, we use the techniques that we have seen so far to think about optimal taxes in dynamic settings. To do so we first have to think about what restrictions we want to put on tax function. In the static setting, this was easy. The government could only see income and there were no markets for insurance against income shocks in advance. Given this, a government's problem becomes

$$\int V(T;\theta)\, dG(\theta)$$

subject to

$$\int T dF(\theta) = g$$
$$V(T;\theta) = \max_y u(y - T(y), y; \theta)$$

Now, we need to turn this into a mechanism design problem. To do so, consider a tax function and the allocation that is implied by it, i.e., $c(\theta; T) = y(\theta; T) - T(y(\theta; T))$ so that $u(c(\theta; T), y(\theta; T); \theta) = V(T; \theta)$. This allocation has to be incentive compatible because

$$\begin{aligned} u\left(y(\theta;T) - T\left(y(\theta;T)\right), y(\theta;T); \theta\right) &\geq u\left(y - T(y), y; \theta\right) \\ &\geq u\left(y\left(\hat{\theta};T\right) - T\left(y\left(\hat{\theta};T\right)\right), y\left(\hat{\theta};T\right); \theta\right) \end{aligned}$$

Moreover, consider an incentive compatible mechanism $c(\theta), y(\theta)$. Note that, by incentive compatbility, if $y(\theta) = y\left(\hat{\theta}\right) (= y)$, then we must have $c(\theta) = c\left(\hat{\theta}\right)$, since

$$\begin{aligned} u\left(c(\theta), y; \theta\right) &\geq u\left(c\left(\hat{\theta}\right), y; \theta\right) \\ u\left(c\left(\hat{\theta}\right), y; \hat{\theta}\right) &\geq u\left(c(\theta), y; \hat{\theta}\right) \end{aligned}$$

This means that a function $T(y)$ must exist such that $c(\theta) = y(\theta) - T(y(\theta))$. What we have proved is the *Taxation Principle.*

In dynamic environments this becomes a bit trickier. It is very plausible to assume that there are some insurance markets to insure households against some of the productivity shocks they receive; health insurance for instance. This means that when we right down the problem of an agent, we should take into account what type of insurance markets are available to the household. Unfortunately, our underlying friction – private information – does not pin down the limits of government and markets. So for now, we assume that markets are incomplete and that the only tool available to households for insurance is self-insurance a la Aiyagari. More instead of writing the problem as a problem of finding best taxes, we start directly from the mechanism design problem and then derive the properties of the tax functions that can implement efficient allocation.

- The setup is as follows – based on Golosov, Kocherlakota, Tsyvinski: time: $t = 0, 1, \cdots, T$ with $T \in \mathbb{N} \cup \{\infty\}$
- A continuum of households subject to productivity shocks: θ_t. Most general stochastic process for θ_t: There exists some distribution $\pi\left(\theta^T\right)$ overall possible histories.
- Preferences:

$$\sum_{t=0}^{T} \beta^t \left[u(c_t) - v\left(\frac{y_t}{\theta_t}\right) \right]$$

- Allocations:

$$\begin{aligned} c_t\left(\theta^T\right) &: \Theta^T \to \mathbb{R} \\ y_t\left(\theta^T\right) &: \Theta^T \to \mathbb{R} \end{aligned}$$

such that $c_t\left(\theta^T\right)$ and $y_t\left(\theta^T\right)$ are both measurable with respect to θ^t – they cannot depend on information that has not arrived at period t.

- Feasible allocations

$$\int c_t\left(\theta^T\right) d\pi\left(\theta^T\right) + K_{t+1} - (1-\delta) K_t = F\left(K_t, \int y_t\left(\theta^T\right) d\pi\left(\theta^T\right)\right)$$

- Incentive compatibility:

$$E_0\left\{\sum_{t=0}^{T} \beta^t \left[u\left(c_t\left(\theta^T\right)\right) - v\left(\frac{y_t\left(\theta^T\right)}{\theta_t}\right)\right]\right\} \geq$$
$$E_0\left\{\sum_{t=0}^{T} \beta^t \left[u\left(c_t\left(\sigma\left(\theta^T\right)\right)\right) - v\left(\frac{y_t\left(\sigma\left(\theta^T\right)\right)}{\theta_t}\right)\right]\right\}, \forall \sigma : \Theta^T \to \Theta^T$$

- Mechanism Design Problem is to maximize agent's ex-ante utility subject to incentive compatibility and feasibility.

Inverse Euler Equation

- A key property that we have come across before is the Inverse Euler Equation. Any solution of the mechanism design problem should satisfy

$$\frac{1}{u'\left(c_t\left(\theta^T\right)\right)} = \frac{1}{\beta\left[1-\delta+F_{K,t+1}\right]} E_t \frac{1}{u'\left(c_{t+1}\left(\theta^T\right)\right)}$$

- The idea behind this equation is that this is an Euler equation for the insurance agency. Consider an insurance agency who wants to decide between whether to deliver utility(from consumption) in period t and history θ^t or to deliver utility in period $t+1$ for all histories θ^{t+1} following θ^t. At the margin this insurance agency should be indifferent to deliver utility at t vs. $t+1$. To see what this implies, consider a small decrease in utility at history θ^t by ε and a small increase in utility from consumption at $t+1$ in all states θ^{t+1} following θ^t, i.e.,

$$\begin{aligned} u\left(\hat{c}_t\left(\theta^T\right)\right) &= u\left(c_t\left(\theta^T\right)\right) - \varepsilon \\ u\left(\hat{c}_{t+1}\left(\theta^T\right)\right) &= u\left(c_{t+1}\left(\theta^T\right)\right) + \beta^{-1}\varepsilon \end{aligned}$$

  Note that this perturbation leaves incentive compatibility unchanged. At θ^t the expected utility of the households, when telling the truth and when lying has not changed. Moreover, at $t+1$, since utility of the households with the common history θ^t has gone up uniformly, neither of them has an additional incentive to lie.

Now, the benefit of the above perturbation at t is given by

$$\frac{\varepsilon}{u'\left(c_t\left(\theta^T\right)\right)}$$

while the cost of it at $t+1$ is given by

$$\frac{1}{\beta}E_t\frac{\varepsilon}{u'\left(c_{t+1}\left(\theta^T\right)\right)}$$

To make the benefit in terms of period t goods, we have to discount them by the gross interest rate given by $1-\delta+F_{K,t+1}$ and hence we have the above equation.

- Using Jensen's inequality, we have that

$$u'\left(c_t\left(\theta^T\right)\right)\leq\beta\left[1-\delta+F_{K,t+1}\right]E_t u'\left(c_{t+1}\left(\theta^T\right)\right)$$

with equality only if there is no variation in $c_{t+1}\left(\theta^T\right)$, or the household should be saving constrained. Later we will interpret this as a tax on capital.

- To see more intuitively why the above equation holds, we consider the following simple example:
- 2-periods, $\theta_0=\bar{\theta}$ and $\theta_1\in\{\theta_L<\theta_H\}$ with $\pi_H=\Pr\left(\theta_1=\theta_H\right)=1-\pi_L$.
- $F\left(K,Y\right)=RK+Y$ and $\delta=1$.
- So the mechanism design problem can be written as

$$\max u\left(c_0\right)-v\left(\frac{y_0}{\bar{\theta}}\right)+\sum\pi_i\left[u\left(c_{1i}\right)-v\left(\frac{y_{1i}}{\theta_i}\right)\right]$$

subject to

$$\begin{aligned}c_0+\frac{1}{R}\sum_i\pi_i c_{1i} &\leq y_0+\frac{1}{R}\sum_i\pi_i y_{1i}\\ u\left(c_{1L}\right)-v\left(\frac{y_{1L}}{\theta_H}\right) &\leq u\left(c_{1H}\right)-v\left(\frac{y_{1H}}{\theta_H}\right)\end{aligned}$$

Now take the solution to the above problem $\left\{c_0,y_0,\{c_{1i},y_{1i}\}_{i=H,L}\right\}$. There are two key properties that will provide intuition for us. First, $c_{1H}>c_{1L}$. Second, the IC constraint binds with equality.

- Now consider increasing the household's risk-free saving at date 0 by $\varepsilon>0$. In addition to the usual cost and benefit of this saving – absent private information this leads to the usual Euler Euqation, this saving has an effect on the incentives for truth-telling in the future. The RHS of the IC constraint is increased by $u'(c_{1H}) R\varepsilon$ and its LHS is increased by $u'(c_{1L}) R\varepsilon$. Now since $c_{1H} > c_{1L}$ and u is concave, the RHS is increased by less than the LHS. Hence saving decreases the incentive for truth-telling in the future. Hence, in addition to decreasing consumption today, saving tightens future incentive constraints and therefore it has an additional cost. Hence, on the margin

$$u'(c_0) + \text{Incentive Cost of saving } = RE_0 u'(c_1)$$

This implies that the household should be saving constrained.

- An important assumption for this result is the separability of consumption and leisure. To see how different specifications determine whether the household should be saving constrained or borrowing constrained, consider the following alteration to the example above: preferences in the second period are given by $U\left(c_1 - v\left(\frac{y_1}{\theta}\right)\right)$. Under this, the incentive constraint becomes

$$U\left(c_{1L} - v\left(\frac{y_{1L}}{\theta_H}\right)\right) \leq U\left(c_{1H} - v\left(\frac{y_{1H}}{\theta_H}\right)\right)$$

  Since this incentive constraint binds in the second period, the solution to the mechanism design problem must satisfy

$$c_{1L} - v\left(\frac{y_{1L}}{\theta_H}\right) = c_{1H} - v\left(\frac{y_{1H}}{\theta_H}\right)$$

  Now consider increasing saving by ε and increasing c_{1L} and c_{1H} by $R\varepsilon$. As it can be seen from above, this perturbation changes the RHS and LHS of the above IC by the same amount. Hence, the incentive cost of saving in this model is zero and therefore Euler Equation must hold

$$u'(c_0) = RE_0 U'(c_1 - v(y_1/\theta_1))$$

- As the above analysis suggests, whether the houehold should be saving constrained or borrowing constrained, depends on how marginal utility when telling the truth is compared to marginal utility when lying for binding incentive constraint. In the separable case, marginal utility when the telling the truth is lower than when lying hence saving tightens the incentive constraints in the future. In Shourideh(2011), I develop a model with capital income risk where truth-telling marginal utility is higher than lying marginal utility and hence sometimes it is optimal to subsidize saving.

Implementation

- So far we have focused on the mechanism design problem. Here we try to see, first through the lens of the above example, how to implement the solution to the above mechanism design problem. As mentioned before, we assume that the only asset available to the households is a risk free bond. Now consider the above two period example and suppose that there are no taxes in the first period and taxes in the second period are a function of labor income and asset/capital income. Given this market structure, the household budget constraint are given by

$$
\begin{aligned}
c_0 + b_1 &\leq y_0 \\
c_1 &\leq y_1 + (1 + r_1) b_1 - T(y_1, r_1 b_1)
\end{aligned}
$$

Household's problem is given by

$$
\max u(c_0) - v\left(\frac{y_0}{\bar{\theta}}\right) + \sum_i \pi_i \left[u(c_{1i}) - v(y_{1i}/\theta_i)\right]
$$

subject to

$$
\begin{aligned}
c_0 + b_1 &\leq y_0 \\
c_{1i} &\leq y_{1i} + (1 + r_1) b_1 - T(y_{1i}, r_1 b_1)
\end{aligned}
$$

where $r_1 = R - 1$ is the equilibrium interest rate.

- The goal is to find a tax function $T(\cdot, \cdot)$ so that the solution to the above problem coincides with the solution to the mechanism design problem $c_0^*, y_0^*, c_{1i}^*, y_{1i}^*, b_1^* (= y_0^* - c_0^*)$.
- Let's start from a simple tax function $T(y_1, r_1 b_1) = T^y(y_1) + \tau_k r_1 b_1$. Given the allocation, the candidate for τ_k is given by

$$
\tau_k^* = 1 + \frac{1}{r_1} - \frac{u'(c_0^*)}{r E_0 u'\left(c_1^*\right)}
$$

Therefore, $T^y(y)$ can be constructed from the budget constraint:

$$
T^y(y_{1i}^*) = y_{1i}^* - c_{1i}^* + (1 + r_1(1 - \tau_k^*)) b_1^*
$$

For other values of y, we construct $T^y(y)$ as in the static case;

- Obviously, the solution of the mechanism design problem is feasible. However, it turns out that for this specific tax function, there is a deviation that delivers more utility. Consider the following allocation:

$$\begin{aligned}
c_0 &= y_0^* - \varepsilon \\
b_1 &= b_1^* + \varepsilon \\
y_{1i} &= y_{1L}^* \\
c_{1i} &= y_{1i}^* - T^y\left(y_{1i}^*\right) + \left(1 + r_1\left(1 - \tau_k^*\right)\right)\left(b_1^* + \varepsilon\right) \\
&= c_{1L}^* + \left(1 + r_1\left(1 - \tau_k^*\right)\right)\varepsilon
\end{aligned}$$

The utility from the constrained efficient allocation is given by

$$u\left(c_0\right) - v\left(y_0/\bar{\theta}\right) + \sum_i \pi_i\left[u\left(c_{1L}^*\right) - v\left(y_{1L}^*/\theta_i\right)\right]$$

where we have used the fact that incentive constraint binds.

The utility from the above plan is given by

$$u\left(c_0 - \varepsilon\right) - v\left(y_0/\bar{\theta}\right) + \sum_i \pi_i\left[u\left(c_{1L}^* + \varepsilon\left(1 + r\left(1 - \tau_k^*\right)\right)\right) - v\left(y_{1L}^*/\theta_i\right)\right]$$

The change in utility is approximately equal to

$$\begin{aligned}
&-u'\left(c_0\right)\varepsilon + \sum_i \pi_i u'\left(c_{1L}^*\right)\left(1 + r\left(1 - \tau_k^*\right)\right)\varepsilon \\
&\quad > -u'\left(c_0\right)\varepsilon + \sum_i \pi_i u'\left(c_{1i}^*\right)\left(1 + r\left(1 - \tau_k^*\right)\right)\varepsilon = 0
\end{aligned}$$

The idea behind this deviation is that when facing a linear tax function on saving, the household finds it optimal to save a bit more than what she is supposed to and work less.

- One way to resolve this issue is to allow for non-linear asset/capital income tax. For example, suppose that the capital tax function has a kink at $r_1 b_1^*$. That is

$$T\left(y_1, r_1 b_1\right) = T^y\left(y_1\right) + T^b\left(r_1 b_1\right)$$

with

$$\begin{aligned}
\lim_{b_1 \searrow b_1^*} T^{b\prime}\left(r_1 b_1\right) &= 1 + \frac{1}{r} - \frac{u'\left(c_0^*\right)}{r u'\left(c_{1L}^*\right)} \\
\lim_{b_1 \nearrow b_1^*} T^{b\prime}\left(r_1 b_1\right) &= 1 + \frac{1}{r} - \frac{u'\left(c_0^*\right)}{r E u'\left(c_{1L}^*\right)}
\end{aligned}$$

Then clearly the above double deivation doesn't not deliver higher utility. However, there might be some other deviations that deliver a higher level of utility. To find the tax function $T^b\left(r_1 b_1\right)$ we do as follows:

Consider the following maximization problem

$$V(x) = \max_{y_{1i}, c_{1i}} \sum_i \pi_i \left[u(c_{1i}) - v(y_{1i}/\theta_{1i}) \right]$$

subject to

$$c_{1i} = y_{1i} - T^y(y_{1i}) + x$$

Now, define the function $T^b(r_1 b_1)$:

$$u(y_0 - b_1) - v(y_0/\bar{\theta}) + V\left((1 + r_1)b_1 - T^b(b_1 r_1)\right) = u^*$$

where u^* is the utility level from the solution to the mechanism design problem. This is the lowest asset tax function that can implement the optimal allocation.

See Werning(2011, 'Nonlinear Capital Taxation') for a proof of differentiability of $T^b(r_1 b_1)$ when the distribution of shocks is continuous as well as its extension of multiple periods.

- There are other implementations. In particular implementations where the capital/asset income taxes are risky a la Kocherlakota(2005) and Albanesi and Sleet(2006), i.e., the tax function is non-separable between labor and capital income.

i.i.d. Shocks

- Here we study optimal capital and labor income taxes in a dynamic setting with i.i.d. shocks. As before, the problem can be written in recursive form using promise utility w as state variable. The mechanism design problem as before is given by

$$P_t(w) = \min \int \left[c(\theta) - y(\theta) + q_{t+1} P_{t+1}\left(w'(\theta)\right) \right] dF(\theta)$$

subject to

$$\begin{aligned} \int \left[u(c(\theta)) + h(y(\theta), \theta) + \beta w'(\theta) \right] dF(\theta) &= w \\ u(c(\theta)) + h(y(\theta), \theta) + \beta w'(\theta) &\geq u\left(c(\hat{\theta})\right) + h\left(y(\hat{\theta}), \theta\right) + \beta w'\left(\hat{\theta}\right) \end{aligned}$$

The above formulation is more general than before. In particular, it allows us to study cases where the shock is a taste shock to the value of leisure:

$$h(y, \theta) = \theta \hat{h}(1 - y)$$

- Before trying to characterize the solution to the above problem, we first try and see if we can show the convexity of the value function. Obviously at its current form the constraint set of the above problem is not concave – concave functions showing up on the RHS of the IC. This mean that the usual techniques from SLP cannot be applied. However, there is a transformation of variables that might make the constraint set convex. Consider the following variables

$$\begin{aligned} v(\theta) &= u(c(\theta)) \\ \eta(\theta) &= h(y(\theta),\theta) \end{aligned}$$

Note that the period objective is given by

$$C(v(\theta)) - Y(\eta(\theta),\theta)$$

where

$$\begin{aligned} u(C(v)) &= v \\ h(Y(\eta,\theta),\theta) &= \eta \end{aligned}$$

- Under this transformation, the constrain set can be written as

$$\begin{aligned} \int \left[v(\theta) + \eta(\theta) + \beta w'(\theta)\right] dF(\theta) &= w \\ v(\theta) + \eta(\theta) + \beta w'(\theta) &\geq v\left(\hat{\theta}\right) + h\left(Y\left(\eta\left(\hat{\theta}\right),\hat{\theta}\right),\theta\right) + \beta w'\left(\hat{\theta}\right) \end{aligned}$$

For this constraint set to be convex, we must have that $h\left(Y\left(\eta\left(\hat{\theta}\right),\hat{\theta}\right),\theta\right)$ is a convex function of $\eta\left(\hat{\theta}\right)$. In general it is hard to come up with conditions on $h(\cdot,\cdot)$ so that this function is convex. There are some cases, however, that the above function becomes linear in $\eta\left(\hat{\theta}\right)$. These are cases where $h(y,\theta) = \hat{h}(y)\,g(\theta)$ where $\hat{h}(y)$ is concave and decreasing. Under this assumption

$$Y(\eta,\theta) = \hat{h}^{-1}\left(\frac{\eta}{g(\theta)}\right)$$

and hence

$$\begin{aligned} h\left(Y\left(\eta\left(\hat{\theta}\right),\hat{\theta}\right),\theta\right) &= h\left(\hat{h}^{-1}\left(\frac{\eta\left(\hat{\theta}\right)}{g\left(\hat{\theta}\right)}\right),\theta\right) \\ &= \hat{h}\left(\hat{h}^{-1}\left(\frac{\eta\left(\hat{\theta}\right)}{g\left(\hat{\theta}\right)}\right)\right) g(\theta) \\ &= \frac{\eta\left(\hat{\theta}\right)}{g\left(\hat{\theta}\right)} g(\theta) \end{aligned}$$

There are two examples that satisfy the above criterion:

– Productivity shocks + constant Frisch Elasticity specification

$$h(y,\theta) = -\frac{1}{1+\gamma}\left(\frac{y}{\theta}\right)^{1+\gamma}$$

– Taste shock to the value of leisure

$$h(y,\theta) = \theta h(1-y)$$

Here y can be interpted as hours worked which is observable.

- Another way of ensuring that the constraint set is convex is to assume that only downward constraints bind and that $\frac{h_{11}}{h_1}$ is a decreasing function of θ – can be interpreted as decreasing absolute risk aversion with respect to θ.

- Example: Here we focus on an example where we have closed form solutions and see how taxes evolve over time and in the cross section. Assume that preferences have the following form

$$\log c + \frac{1}{\theta}\log(1-y)$$

These preferences imply that optimal static distortions are independent of how much wealth is brought in by the household, i.e., MRS between c and l is the same when telling the truth and when lying.

- The recursive formulation is given by

$$P_t(w) = \min \int \left[c(\theta) - y(\theta) + q_{t+1}P_{t+1}\left(w'(\theta)\right)\right] dF(\theta)$$

subject to

$$\begin{aligned}
\int U(\theta)\, dF(\theta) &= w \\
\log c(\theta) + \theta^{-1}\log(1-y(\theta)) + \beta w'(\theta) &= U(\theta) \\
\log c(\theta) + \theta^{-1}\log(1-y(\theta)) + \beta w'(\theta) &\geq \log c\left(\hat{\theta}\right) + \theta^{-1}\log\left(1-y\left(\hat{\theta}\right)\right) + \beta w'\left(\hat{\theta}\right)
\end{aligned}$$

If we let $v(\theta) = \log c(\theta)$ and $\eta(\theta) = \log(1-y(\theta))$, then we can rewrite this problem as

$$P_t(w) = \min \int \left[e^{v(\theta)} + e^{\eta(\theta)} - 1 + q_{t+1}P_{t+1}\left(w'(\theta)\right)\right] dF(\theta)$$

subject to

$$\begin{aligned}
\int U(\theta)\, dF(\theta) &= w \\
v(\theta) + \theta^{-1}\eta(\theta) + \beta w'(\theta) &= U(\theta) \\
U'(\theta) &= -\theta^{-2}\eta(\theta)
\end{aligned}$$

Starting from last period, $t = T$, we can guess that

$$\begin{aligned}
v_T(\theta, w) &= \hat{v}_T(\theta) + \frac{w}{1 + E\theta^{-1}} \\
\eta_T(\theta, w) &= \hat{\eta}_T(\theta) + \frac{w}{1 + E\theta^{-1}} \\
P_T(w) &= e^{\frac{w}{1+E[1/\theta]}} \int \left[e^{\hat{v}_T(\theta)} + e^{\hat{\eta}_T(\theta)} \right] dF(\theta) - 1 \\
&= A_T e^{\frac{w}{1+E[1/\theta]}} - 1
\end{aligned}$$

where $\{\hat{\eta}_T(\theta)\}$ and $\{\hat{v}_T(\theta)\}$ are the solution to the planning problem

$$\min_{\{v(\theta), \eta(\theta)\}} \int \left[e^{v(\theta)} + e^{\eta(\theta)} \right] dF(\theta)$$

subject to

$$\begin{aligned}
\int [v(\theta) + \theta\eta(\theta)]\, dF(\theta) &= 0 \\
v'(\theta) + \theta^{-1}\eta'(\theta) &= 0
\end{aligned}$$

Note that given this solution, labor wedge is given by

$$\begin{aligned}
1 - \tau_{l,T}(\theta) &= \frac{h'(1 - y(\theta))}{\theta u'(c(\theta))} \\
&= \frac{1}{\theta} e^{\hat{v}_T(\theta) - \hat{\eta}_T(\theta)}
\end{aligned}$$

and hence is independent of history so far.

- At $T - 1$, the recursive formulation of the problem is given by

$$P_{T-1}(w) = \min \int \left[e^{v(\theta)} + e^{\eta(\theta)} - 1 + q_T \left(A_T e^{\frac{w'(\theta)}{1+E[1/\theta]}} - 1 \right) \right] dF(\theta)$$

subject to

$$
\begin{aligned}
\int U(\theta)\, dF(\theta) &= w \\
v(\theta) + \theta^{-1}\eta(\theta) + \beta w'(\theta) &= U(\theta) \\
U'(\theta) &= -\theta^{-2}\eta(\theta)
\end{aligned}
$$

Note that since consumption is separable from labor supply, th margin between c and w' is undistorted, i.e.,

$$e^{v(\theta)} = \beta^{-1} q_T A_T \frac{1}{1 + E[1/\theta]} e^{\frac{w'(\theta)}{1+E[1/\theta]}}$$

Hence, the objective can be written as

$$P_{T-1}(w) = \min \int \left[e^{v(\theta)} \left(1 + \beta \left(1 + E[1/\theta]\right)\right) + e^{\eta(\theta)} \right] dF(\theta) - 1 - q_T$$

subject to

$$
\begin{aligned}
\int U(\theta)\, dF(\theta) &= w \\
v(\theta) + \theta^{-1}\eta(\theta) + \beta w'(\theta) &= U(\theta) \\
U'(\theta) &= -\theta^{-2}\eta(\theta)
\end{aligned}
$$

Policy functions:

$$
\begin{aligned}
v_{T-1}(w,\theta) &= \hat{v}_{T-1}(\theta) + \frac{w}{(1+\beta)\left(1+E[1/\theta]\right)} \\
\eta_{T-1}(w,\theta) &= \hat{\eta}_{T-1}(\theta) + \frac{w}{(1+\beta)\left(1+E[1/\theta]\right)} \\
w_{T-1}(w,\theta) &= \hat{w}_{T-1}(\theta) + \frac{w}{1+\beta} \\
P_{T-1}(w) &= A_{T-1} e^{\frac{w}{(1+\beta)(1+E[1/\theta])}}
\end{aligned}
$$

where $\{\hat{v}_{T-1}(\theta), \hat{\eta}_{T-1}(\theta)\}$ solves the following problem

$$A_{T-1} = \min \int \left[e^{v(\theta)} \left(1 + \beta \left(1 + E[1/\theta]\right)\right) + e^{\eta(\theta)} \right] dF(\theta)$$

subject to

$$
\begin{aligned}
\int \left[v(\theta) \left(1 + \beta \left(1 + E[1/\theta]\right)\right) + \theta^{-1}\eta(\theta) \right] dF(\theta) &= \beta \left(1 + E[1/\theta]\right) \log \left[\beta^{-1} q_T A_T / \left(1 + E[1/\theta]\right) \right] \\
v'(\theta) \left(1 + \beta \left(1 + E[1/\theta]\right)\right) + \theta^{-1}\eta'(\theta) &= 0
\end{aligned}
$$

- In general, we can show that

$$\begin{aligned}
v_t(w,\theta) &= \hat{v}_t(\theta) + \frac{w}{\alpha_t} \\
\eta_t(w,\theta) &= \hat{\eta}_t(\theta) + \frac{w}{\alpha_t} \\
w'_t(w,\theta) &= \alpha_{t+1} v_t(w,\theta) - \psi_t \\
P_t(w) &= A_t e^{\frac{w}{\alpha_t}}
\end{aligned}$$

  where ψ_t is some constant, $\alpha_t = \left(1 + \beta + \cdots + \beta^{T-t}\right)\left(1 + E\left[1/\theta\right]\right)$, and $\{\hat{v}_t(\theta), \hat{\eta}_t(\theta)\}$ is the solution to the following program

$$A_t = \min \int \left[\left(1 + \beta\alpha_{t+1}\right) e^{v(\theta)} + e^{\eta(\theta)}\right] dF(\theta)$$

  subject to

$$\begin{aligned}
\int \left[\left(1 + \beta\alpha_{t+1}\right) v(\theta) + \theta^{-1}\eta(\theta)\right] dF(\theta) &= \beta\psi_t \\
\left(1 + \beta\alpha_{t+1}\right) v'(\theta) + \theta^{-1}\eta'(\theta) &= 0
\end{aligned}$$

  From this, we can see that the labor wedge

$$\begin{aligned}
1 - \tau_{l,t}\left(\theta^t\right) &= \frac{h'\left(1 - l_t\left(\theta^t\right)\right)}{\theta_t u'\left(c_t\left(\theta^t\right)\right)} \\
&= \frac{c_t\left(\theta^t\right)}{\theta_t\left(1 - l_t\left(\theta^t\right)\right)} = \frac{1}{\theta_t} e^{\hat{v}_t(\theta_t) - \hat{\eta}_t(\theta_t)}
\end{aligned}$$

  Hence, labor wedges are indpendent of history. However, they depend on t. Moreover, taxes should increase over life cycle.

- What about capital taxes