Mechanism Design - Problem Set 1

March 12, 2021

1. **Auctions with Discrete Types.** In class, we showed that when the type distribution is continuous, revenue equivalence holds. Verify whether this is true when the type space is discrete.

2. **Optimal Taxation.** In class, we describe the problem of optimal taxation for a government that faces a population of workers (consisted of a unit continuum of workers. Note that we can allude to a version of law of large number and assume that this is equivalent to having one worker) with heterogeneous productivities, $\theta \in \mathbb{R}_+$, whose preferences over consumption and income are given by $c - v\left(y/\theta\right)$ with $v\left(l\right) = \psi l^{\gamma}/\gamma$ for $\gamma > 1, \psi > 0$. Suppose that $\theta \sim F\left(\theta\right)$ has a continuous distribution with $F\left(\cdot\right)$ as its c.d.f. The government is interested in choosing a non-linear tax function $T\left(\cdot\right)$ and workers choose how much income to earn while their productivity is only privately known to them. Suppose that the objective of the government is to maximize revenue from taxation.

    1. Define a direct mechanism.
    2. Use the revelation principle to cast the problem as a mechanism design problem.
    3. Show that any direct mechanism can be implemented with a tax function. This is often referred to as the taxation principle.
    4. Use the envelope formulation to simplify the incentive constraints.
    5. Use optimization technique introduced in lecture 1 to solve for optimal direct mechanism. *Hint: Ignore the monotonicity constraint.*
    6. Suppose that $\text{Supp}\left(F\right) = \left[\underline{\theta}, \overline{\theta}\right]$. What is optimal marginal tax rate for the most productive worker?
    7. Now suppose that $\text{Supp}\left(F\right) = \left[\underline{\theta}, \infty\right]$. What is the optimal marginal tax rate as $\theta \to \infty$?

3. **Delegation.** In a lot of economic settings of interest, contracting must be done without the use of monetary transfers. For example, consider the interaction between an employer (principal, P) and an employee (agent, A). Suppose that A can take an observable action $a \in [0, 2]$. The payoff for P from the action is $-\left(a - \theta\right)^2$ where $\theta \in [1, 2]$ is the state of the world. Suppose that $\theta \sim U\left[1, 2\right]$. On the other hand, suppose that A has a payoff of $-\left(a - \theta + b\right)^2$ for some

$b \in (0, 1)$. The key assumption is that A knows the state while the employer does not. The choice of P is which actions to allow A to take. In other words, P chooses an action set $M \subset \mathbb{R}_+$ while A is free to choose whatever action she wants within the set M.

1. Use the relevation principle to cast the problem of P – finding the set M to maximize her expected payoff – as a direct mechanism design problem.
2. Similar to the auction model, simplify A's incentive constraint using an envelope and monotonicity condition.
3. Give two examples of incentive compatible mechanisms? How do you implement these using delegation sets?
4. Now focus on mechanisms that maximize the payoff of P. What can you say about the payoff of A when $\theta = 1$? Prove your statement.
5. Prove that the solution of P's problem is of the form $a(\theta) = 1, \theta \leq \theta^*$ and $a(\theta) = \theta - b$ for $\theta \geq \theta^*$. Find θ^*.

4. Revelation Principle and Commitment. Consider a monopolist selling a good to a buyer whose wants to purchase multiple units of the good and whose payoff is given by $\theta q - p$ where $\theta \in \{1, 2\}$ with $\Pr(\theta = 2) = 1/2$. Suppose that the seller's cost of production is $\frac{1}{2}q^2$. The seller does not know θ while the buyer does. Buyer has an outside option of 0.

1. Suppose that the seller and the buyer play the following game: The seller offers a menu of options, i.e., prices and quantities, to the buyer and the buyer chooses her appropriate option. Use revelation principle to case this problem as a direct revelation mechanism.
2. Find the direct revelation mechanism that maximizes seller's profit.
3. Now consider an alternative game between the seller and the buyer: The seller announces a menu, the buyer makes her choice. After that the seller has the option to revise her offer at a cost c. Show that for c small enough, you cannot use the revelation principle to describe any outcome in the resulting game.
4. In this environment, how would you modify the revelation principle in order to describe the set of arbitrary outcomes of this game.
5. What is the optimal value of profit in this case?