Problem 1

Part (a)

Assume that the budget constraint holds with equality. Then we can write this problem as

$$\sup_{\{k_{t+1},n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\big(f(k_t,n_t)+(1-\delta)k_t-k_{t+1},1-n_t\big) \ s.t.$$
$$n_t \in [0,1]$$
$$k_{t+1} \in [0, f(k_t,n_t)-(1-\delta)k_t] \tag{1}$$

Note that from the problem's assumptions, $k_{t+1}=0$ is possible; households can always consume the $(1-\delta)k_t$ in savings left over from previous periods.

Part (b)

Writing (1) in recursive form:

$$V(k)=\max_{k'} \big[u\big(f(k,n)+(1-\delta)k-k',1-n\big)+\beta V(k')\big] \ \ s.t.$$
$$n_t \in [0,1]$$
$$k' \in [0, f(k,n)-(1-\delta)k] \tag{2}$$

For the solution to this problem to be the same as the solution to the sequence problem, we require four assumptions to hold:

1. $\Gamma(k)$ is non-empty for all $k \in K$ where K is the set of feasible capital
2. For all feasible plans $\hat{k}(k_0)$, $\lim_{T\to\infty}\sum_{t=0}^{T}\beta^t u(\hat{k}_t,\hat{k}_{t+1},1-n_t)$ exists
3. For any plan $\bar{k}(k_0)$ satisfying (2), $\lim_{t\to\infty}\sup\beta^t V(\bar{k}_t)\le 0$

To satisfy the first assumption, we require that $1-\delta\ge 0$ and $f(k,n)\ge 0$ so that $k_{t+1}=0$ is always possible. The second assumption is satisfied given that the utility and production functions are continuous over $\mathbb{R}^+$. The third assumption will hold provided that $\beta<1$ and $V(k)$ is bounded above (or at least, bounded over the relevant part of $\mathbb{R}^+$). We can then make use of theorems 4.3 and 4.5 from Stokey & Lucas:

Theorem 4.3: if V solves (2) then V is a solution to (1)

Theorem 4.5: if $\bar{k}(k_0)$ is a plan solving (2), then $\bar{k}(k_0)$ maximizes (1)

Part (c)

With these functional form assumptions, we can write the Bellman equation as

$$V(k) = \max_{k',n} \left[\gamma \log\left(k^{\alpha} n^{1-\alpha} - k'\right) + (1-\gamma)\log\left(1-n\right) + \beta V(k')\right] \ \ s.t.$$
$$n_t \in [0,1]$$
$$k' \in [0, k^{\alpha} n^{1-\alpha}]$$

We guess that $V(k) = A + B\log(k)$. Substituting this into the Bellman equation,

$$A + B\log k = \max_{k',n} \left[\gamma \log\left(k^{\alpha} n^{1-\alpha} - k'\right) + (1-\gamma)\log\left(1-n\right) + \beta(A + B\log k')\right]$$

Taking FOC,

$$k': \ \frac{\gamma}{k^{\alpha} n^{1-\alpha} - k'} = \frac{\beta B}{k'}$$
$$n: \ \frac{\gamma(1-\alpha)k^{\alpha} n^{-\alpha}}{k^{\alpha} n^{1-\alpha} - k'} = \frac{1-\gamma}{1-n}$$

Rewriting,

$$k' = \frac{\beta B k^{\alpha} n^{1-\alpha}}{\gamma + \beta B}$$
$$n = \frac{(1-\alpha)(\gamma + \beta B)}{(1-\gamma) + (1-\alpha)(\gamma + \beta B)}$$

Substituting the policy function for capital into the Bellman equation,

$$\begin{aligned}
A + B\log k &= \gamma \log\left(k^{\alpha} n^{1-\alpha} - \frac{\beta B k^{\alpha} n^{1-\alpha}}{\gamma + \beta B}\right) + (1-\gamma)\log\left(1-n\right) + \beta\left(A + B\log \frac{\beta B k^{\alpha} n^{1-\alpha}}{\gamma + \beta B}\right) \\
&= \gamma \log\left(\frac{\gamma k^{\alpha} n^{1-\alpha}}{\gamma + \beta B}\right) + (1-\gamma)\log\left(1-n\right) + \beta\left(A + B\log \frac{\beta B k^{\alpha} n^{1-\alpha}}{\gamma + \beta B}\right) \\
&= \gamma \log\left(\frac{\gamma n^{1-\alpha}}{\gamma + \beta B}\right) + (1-\gamma)\log\left(1-n\right) + \beta\left(A + B\log \frac{\beta B n^{1-\alpha}}{\gamma + \beta B}\right) + \gamma\alpha\log k + \beta B\alpha\log k \\
&= \gamma \log\left(\frac{\gamma}{\gamma + \beta B}\right) + (1-\gamma)\log(1-n) + \beta A + \beta B\log\left(\frac{\beta B}{\gamma + \beta B}\right) \\
&\quad + (\gamma + \beta B)\log\left(n^{1-\alpha}\right) + \alpha(\gamma + \beta B)\log k
\end{aligned}$$

We can now solve for A and B as well as the policy functions for labor and capital:

$$\begin{aligned}
B &= \alpha(\gamma + \beta B) \\
&= \frac{\alpha\gamma}{1-\alpha\beta} \\
n^* &= \frac{\gamma(1-\alpha)}{\gamma(1-\alpha) + (1-\gamma)(1-\alpha\beta)} \\
k'^* &= \beta\alpha k^\alpha n^{1-\alpha} \\
A &= \frac{1}{1-\beta}\left[\gamma \log\left(\frac{\gamma}{\gamma+\beta B}\right) + (1-\gamma)\log(1-n) + \beta B \log\left(\frac{\beta B}{\gamma + \beta B}\right) + (\gamma + \beta B)(1-\alpha)\log n\right] \\
&= \frac{1}{1-\beta}\left[\gamma \log(1-\alpha\beta) + (1-\gamma)\log(1-n) + \frac{\beta\alpha\gamma}{1-\alpha\beta}\log(\beta\alpha) + \frac{\gamma(1-\alpha)}{1-\alpha\beta}\log n\right] \\
&= \frac{1}{1-\beta}\Big[\gamma\log(1-\alpha\beta) + \frac{\beta\alpha\gamma}{1-\alpha\beta}\log(\beta\alpha) + \frac{\gamma(1-\alpha)}{1-\alpha\beta}\log(\gamma[1-\alpha]) + (1-\gamma)\log([1-\gamma][1-\alpha\beta]) \\
&\quad - \frac{1-\alpha\beta+\gamma\alpha\beta-\gamma\alpha}{1-\alpha\beta}\log\big(\gamma(1-\alpha) + (1-\gamma)(1-\alpha\beta)\big)\Big] \\
&= \frac{1}{1-\beta}\Big[\log(1-\alpha\beta) + \frac{\beta\alpha\gamma}{1-\alpha\beta}\log(\beta\alpha) + \frac{\gamma(1-\alpha)}{1-\alpha\beta}\log(\gamma[1-\alpha]) + (1-\gamma)\log(1-\gamma) \\
&\quad - \frac{1-\alpha\beta+\gamma\alpha\beta-\gamma\alpha}{1-\alpha\beta}\log\big(\gamma(1-\alpha) + (1-\gamma)(1-\alpha\beta)\big)\Big]
\end{aligned}$$

Finally, we can solve for optimal consumption:

$$\begin{aligned}
c^* &= \gamma k^\alpha n^{*1-\alpha} - k'^* \\
&= (1-\beta\alpha)k^\alpha n^{*1-\alpha} \\
&= (1-\beta\alpha)k^\alpha\left(\frac{\gamma(1-\alpha)}{\gamma(1-\alpha)+(1-\gamma)(1-\alpha\beta)}\right)^{1-\alpha}
\end{aligned}$$

Part (d)

With the assumption that $\gamma = 1$, we can write the Bellman equation as

$$V(k) = \max_{k'}\left[\log\left(k^\alpha - k'\right) + \beta V(k')\right]$$

which gives us the operator

$$\begin{aligned}
TV(k) &= \max_{k'}\left[\log\left(k^\alpha - k'\right) + \beta V(k')\right] \\
\implies V_{n+1} &= \max_{k'}\left[\log\left(k^\alpha - k'\right) + \beta V_n\right]
\end{aligned}$$

Starting from an initial value $V_0 = 0$, we have

$$\begin{aligned}
V_1 &= \max_{k'}\log(k^\alpha - k') \\
&= \alpha\log(k)
\end{aligned}$$

Taking a second iteration,

$$\begin{aligned}
V_2 &= \max_{k'} \left[\log(k^\alpha - k') + \beta\alpha \log(k')\right] \\
&= \left[\log\left(k^\alpha - \frac{\beta\alpha k^\alpha}{1+\beta\alpha}\right) + \beta\alpha \log\left(\frac{\beta\alpha k^\alpha}{1+\beta\alpha}\right)\right] \\
&= \beta\alpha \log(\beta\alpha) - (1+\beta\alpha)\log(1+\beta\alpha) + \alpha(1+\beta\alpha)\log k
\end{aligned}$$

Define

$$\begin{aligned}
A_1 &= \beta\alpha \log(\beta\alpha) - (1+\beta\alpha)\log(1+\beta\alpha) \\
B_1 &= \alpha(1+\beta\alpha)
\end{aligned}$$

Taking a third iteration,

$$\begin{aligned}
V_3 &= \max_{k'} \left[\log(k^\alpha - k') + \beta[A_1 + B_1\alpha \log k']\right] \\
&= \log\left(k^\alpha - \frac{\beta B_1 \alpha k^\alpha}{1+\beta B_1 \alpha}\right) + \beta\left[A_1 + B_1 \log\left(\frac{\beta B_1 \alpha k^\alpha}{1+\beta B_1 \alpha}\right)\right] \\
&= \beta A_1 + \beta B_1 \log(\beta B_1) - (1+\beta B_1)\log(1-\beta B_1) + \alpha(1+\beta B_1)\log k
\end{aligned}$$

Now define

$$\begin{aligned}
A_2 &= \frac{1}{1-\beta}\left(\beta B_1 \log(\beta B_1) - (1+\beta B_1)\log(1+\beta B_1)\right) \\
B_2 &= \alpha + \beta\alpha B_1
\end{aligned}$$

If we continued iterating, we would have a sequence B_n where

$$B_n = \alpha(1 + \alpha\beta + \alpha^2\beta^2 + ... + \alpha^n\beta^n B_0)$$

where $B_0 = 1$ from above. Under the earlier assumption that $\beta\alpha < 1$, we have the geometric series

$$\begin{aligned}
\lim_{n\to\infty} B_n &= \frac{\alpha}{1-\beta\alpha} \\
\lim_{n\to\infty} A_n &= \frac{1}{1-\beta}\left(\frac{\beta\alpha}{1-\beta\alpha}\log\left(\frac{\beta\alpha}{1-\beta\alpha}\right) + \frac{1}{1-\beta\alpha}\log(1-\beta\alpha)\right) \\
&= \frac{1}{1-\beta}\left(\frac{\beta\alpha}{1-\beta\alpha}\log(\beta\alpha) + \log(1-\beta\alpha)\right)
\end{aligned}$$

which yields the same results as part (c).

Problem 2

From the solutions manual to Stokey & Lucas:

Parts (a)-(c)

We need to establish assumptions 4.1-4.8:

A4.1: Let $\Gamma(x) = [0, f(x)]$. Since by T2 $f(0) = 0, 0 \in \Gamma(x)$ for all x, hence $\Gamma(x)$ is non-empty for all x.

A4.2: Let $F(x_t, x_{t+1}) = U(f(x_t) - x_{t+1})$. By U3 and the fact that $U : \mathbb{R}^+ \to \mathbb{R}$, U, and hence F, is bounded below and the result follows from U1.

A4.3: $X = [0, \bar{x}] \in \mathbb{R}^+$ which is a convex subset of $\mathbb{R}$. $\Gamma(x)$ is non-empty from above. Given x, $[0, f(x)]$ is compact and $\Gamma(x)$ is compact-valued. Since $f(x)$ is continuous, the correspondence $[0, f(x)]$ is continuous.

A4.4: We showed that F is bounded below. By T1-T3, $f(x_t) - x_{t+1}$ is bounded, and hence by assumption U2 and U3, f is bounded above. Hence f is bounded. It is continuous by U2 and T1. By U1, $0 < \beta < 1$.

A4.5: By U3 and T3, $F(\cdot, y)$ is a strictly increasing function.

A4.6: Let $x \leq x'$. Then by T3, $f(x) \leq f(x')$, which implies that $[0, f(x)] \subseteq [0, f(x')]$.

A4.7: By T4 $f(x) - y$ is a concave function in (x, y). By U4 this implies that $F(x, y)$ is strictly concave in (x, y).

A4.8: Let $x, x' \in X, y \in \Gamma(x)$ and $y' \in \gamma(x')$. Then $y \leq f(x)$ and $y' \leq f(x')$, which implies, by T4, that

$$\begin{aligned} \theta y + (1 - \theta)y' &\leq \theta f(x) + (1 - \theta)f(x') \\ &\leq f(\theta x + [1 - \theta]x') \end{aligned}$$

Part (d)

$v(x)$ is differentiable at x: By Theorems 4.7 and 4.8 and parts (b) and (c), v is an increasing and strictly concave function. By U5, T5 and $g(x) \in (0, f(x))$, Assumption 4.9 is satisfed. Hence by Theorem 4.11 v is continuously differentiable and

$$v'(x) = F_x[f(x) - g(x)] = U'[f(x) - g(x)]f'(x)$$

A sufficient condition for the interior solution $g(x) \in (0, f(x))$ is

$$\lim_{c \to 0} U'(x) = \infty$$

To see that $g(x) = f(x)$ is never optimal we have

$$\lim_{g(x) \to f(x)} U'(f(x) - g(x)) = \infty$$

while

$$\lim_{g(x) \to f(x)} \beta v'[g(x)] < \infty$$

by the strict concavity of v. Hence the utility from decreasing x today increases much faster than it decreases the value from saving it. Therefore it cannot be $g(x) = f(x)$. To see that $g(x) = 0$ is not optimal, assume $g(\hat{x}) = 0$ for some $\hat{x} > 0$. Hence it must be that $g(x) = 0$ for all $x < \hat{x}$. But then, for $x < \hat{x}$,

$$v(x) \equiv U(f[x]) + \beta \frac{U(0)}{1 - \beta}$$

Therefore v is differentiable and

$$v'(x) = U'[f(x)]f'(x)$$

Hence, when $x \to 0$, $v'(x) \to 1$, and then $g(\hat{x}) = 0$ for $\hat{x}$ is not possible. To see what happens when this condition fails, notice that at the steady state, we have $g(x^*) < f(x^*)$, where x^* stands for the steady state level of capital. By continuity of g, there is an interval $(x^* \epsilon, x^*)$ such that for any x belonging to that interval, $g(x) < f(x)$. Theorem 4.11 implies that v is differentiable in this range. For any other x, eventually this interval will be reached, or another point interval that implies $g(x) = 0$ or $g(x) < f(x)$. We established above that v is differentiable at those cases, so it must be that v is differentiable everywhere.

Part (e)

Let $\beta' > \beta$. Define T' as the operator T using β' as a discount factor instead of β, and v_k as the k-th application of this operator. Applying T' to $v(x, \beta)$ once we obtain $v_1(x, \beta')$ and $g_1(x, \beta')$, where using first order conditions (assuming an interior solution for simplicity), $g_(x, \beta')$ is defined as the solution y to

$$U'(f[x] - y) = \beta' v(y, \beta')$$

It is clear that the savings function must increase since the right hand side increases from β to β', that is $g_1(x,\beta') > g_1(x,\beta)$, which by Theorem 4.11 in turn implies that

$$v_1'(x,\beta') > v'(x,\beta)$$

By a similar argument, if

$$v_k'(x,\beta') > v_{k-1}'(x,\beta')$$

then

$$v_{k+1}'(x,\beta') > v_k'(x,\beta')$$

and

$$g_{k+1}(x,\beta') > g_k(x,\beta')$$

Hence $g_k(x,\beta')$ increases with k. The result then follows from applying Theorem 4.9 to the sequence $\{g_k(x,\beta')\}_{k=0}^{\infty}$ since $g_k(x,\beta') \to g(x,\beta')$.

Problem 3

See Hakki's solutions last year for a proof that relies on the continuous time formulation of the problem. Here I'll try to present his proof in more economic terms. Consider the production function

$$Y_t = F(A_t K_t, L_t)$$

Suppose the economy is on a balanced growth path, where population grows at rate n and capital and GDP grow at rate g. Technology is capital-augmenting as above; assume it grows at rate g_a. Note that at any BGP, it must be that Y is constant returns to scale, since otherwise we could not have output growing at the same rate as input. Hence we can divide through the production function by L_t:

$$y_t = f(A_t k_t, 1)$$

in which case labor is now constant and output/capital grow at rate $g_y = g - n$. We have growth of per capita output on the left-hand side, and growth in per capita capital and technology on the right-hand side, which by assumption enter the production function together. We can write this relationship in terms of growth rates:

$$\text{\% change in y} = (\text{\% change in } f \text{ given \% change in } Ak) \times (\text{\% change in } Ak)$$
$$\implies g_y = \frac{\delta f(A_t k_t, 1)}{\delta Ak} \frac{A_t k_t}{f(A_t k_t, 1)} (g_a + g_y)$$

and hence

$$\frac{g_y}{g_a + g_y} = \frac{\delta f(A_t k_t, 1)}{\delta k} \frac{k}{f(A_t k_t, 1)} \tag{1}$$

The LHS of (1) is constant by assumption. The right-hand side is the elasticity of output with respect to capital (inclusive of technology), which must also be constant. Intuitively, this means that f must be not only CRS (i.e. homogeneous of degree 1 in all inputs), but also homogeneous of degree α in capital - with a partial derivative w.r.t. capital that is homogeneous of degree $\alpha - 1$ - since only in this case will we have

$$\begin{aligned} \frac{\delta f(A_t k_t, 1)}{f(A_t k_t, 1)} \frac{A_t k_t}{\delta AK} &= \frac{\alpha (A_t k_t)^{\alpha - 1}}{(A_t k_t)^{\alpha}} (A_t k_t) \qquad \text{(i.e. as } \delta \to 0) \\ &= \alpha \frac{A_t k_t}{A_t k_t} \\ &= \alpha \end{aligned}$$

i.e. elasticity equal to a constant. Only a Cobb-Douglas (or log-linear) production function satisfies these requirements.

Problem 4

This problem is studied in Greenwood, Hercowitz and Per Krusell (1997) - see the paper if you want to learn more about it.

Part (a)

For this problem I ignore profits (which will be zero at the CE). I assume that households have a fixed endowment of leisure $\bar{e} = 1$ and that leisure is unvalued. A competitive equilibrium is prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$ and allocations $\{c_t^i, x_t^i, k_t^i, y_t^f, n_t^f, k_t^f\}_{t=0}^{\infty}$ such that

1. Households solve the problem

$$\max_{\{c_t^i, x_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \ s.t.$$
$$p_t(c_t^i + x_t^i) \leq r_t k_t^i + w_t \frac{1}{I}$$
$$k_{t+1} = (1-\delta)k_t + q_t x_t$$
$$q_{t+1} = (1+g_q)q_t$$
$$k_{t+1} \geq 0 \ \forall t$$
$$k_0 \text{ given}$$

2. The representative firm maximizes profits:

$$\max_{y_t^f, k_t^f} p_t y_t^f - r_t k_t^f - w_t n_t^f \ s.t.$$
$$y_t^f = z(k_t^f)^{\alpha}(n_t^f)^{1-\alpha}$$

3. Markets clear:

$$\sum_i \left(c_t^i + x_t^i\right) = y_t^f$$
$$1 = n_t^f$$
$$\sum_i k_t^i = k_t^f \qquad \forall \ i$$

Part (b)

First note that on any balanced growth path, both sides of the investment equation

$$k_{t+1} - (1-\delta)k_t = x_t q_t$$

have to grow at the same rate. If $(1+g_k)$ is the growth rate of capital, and $(1+g_x)$ the growth rate of investment, then we have

$$1 + g_k = (1+g_x)(1+g_q)$$

Additionally, from the production function we must have that at any BGP,

$$(1+g_y) = (1+g_k)^{\alpha}$$

Since $\alpha < 1$, it is clearly the case that $g_k > g_y$. Supposing that $g = g_y = g_x$ (i.e. investment and output [and consumption] all grow at the same rate), we have

$$1 + g_k = (1 + g)(1 + g_q)$$
$$1 + g = (1 + g_k)^{\alpha}$$

and hence

$$1 + g = (1 + g_q)^{\frac{\alpha}{1-\alpha}}$$
$$1 + g_k = (1 + g_q)^{\frac{1}{1-\alpha}} \tag{1}$$

Part (c)

To write the planner's problem recursively, we need to first put the model in terms of a steady state, which means dividing out by the growth rates. Denote the following:

$$\hat{c}_t = c_t/(1 + g)^t$$
$$\hat{x}_t = x_t/(1 + g)^t$$
$$\hat{u}_t = y_t/(1 + g)^t$$
$$\hat{k}_t = k_t/(1 + g_k)^t$$
$$\hat{q}_t = q_t/(1 + g_q)^t$$

Then

$$\begin{aligned}
\hat{x}_t &= \frac{k_{t+1} - (1-\delta)k_t}{(1+g_q)^{\frac{t\alpha}{1-\alpha}} q_t} \\
&= \frac{(1+g_q)^{\frac{t+1}{1-\alpha}} \hat{k}_{t+1} - (1+g_q)^{\frac{t}{1-\alpha}} (1-\delta)\hat{k}_t}{(1+g_q)^{\frac{t\alpha}{1-\alpha}} (1+g_q)^t \hat{q}_t} \\
&= \frac{(1+g_q)^{\frac{1}{1-\alpha}} \hat{k}_{t+1} - (1-\delta)\hat{k}_t}{\hat{q}_t} \\
&= \frac{(1+g_q)^{\frac{1}{1-\alpha}} \hat{k}_{t+1} - (1-\delta)\hat{k}_t}{\hat{q}}
\end{aligned}$$

where the last equality follows from the fact that $\hat{q}_{t+1}/\hat{q}_t = 1$. And in yet another example of the wonders of Cobb-Douglas:

$$\begin{aligned}\hat{y}_t &= \frac{zk_t^\alpha}{(1+g_q)^{\frac{t\alpha}{1-\alpha}}} \\ &= \frac{z(1+g_q)^{\frac{t\alpha}{1-\alpha}}\hat{k}_t^\alpha}{(1+g_q)^{\frac{t\alpha}{1-\alpha}}} \\ &= z\hat{k}_t^\alpha\end{aligned}$$

We therefore have the planner's problem

$$\max_{\{\hat{c}_t,\hat{k}_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(\hat{c}_t) \ s.t.$$
$$\hat{c}_t + \frac{(1+g_q)^{\frac{1}{1-\alpha}}}{\hat{q}} k_{t+1} = \frac{1-\delta}{\hat{q}}\hat{k}_t + z\hat{k}_t^\alpha$$

Substituting the feasibility constraint into the utility function and taking the FOC to get the Euler equation:

$$\frac{(1+g_q)^{\frac{1}{1-\alpha}}}{\hat{q}} u_{c,t} = \beta u_{c,t+1}\left(z\alpha\hat{k}_{t+1}^{\alpha-1} + \frac{1-\delta}{\hat{q}}\right)$$

and rewriting,

$$\frac{u_{c,t}}{u_{c,t+1}} = \frac{\beta}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(\hat{q}z\alpha\hat{k}_{t+1}^{\alpha-1} + 1 - \delta\right) \tag{2}$$

If the economy starts with a low level of capital, the RHS of (2) is greater than 1 - in which case the left-hand side is also greater than 1, which implies that consumption is growing (and the economy accumulating capital) given the standard assumption that u is strictly concave. On the other hand, as capital goes towards infinity, we have

$$\lim_{\hat{k}\to\infty} \frac{\beta}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(\hat{q}z\alpha\hat{k}_{t+1}^{\alpha-1} + 1 - \delta\right) = \frac{\beta(1-\delta)}{(1+g_q)^{\frac{1}{1-\alpha}}}$$

which is clearly less than 1, indicating that investment is negative and consumption is declining over time. Hence the economy will eventually converge to a steady state where the LHS of (2) is equal to 1, and at this level we will have

$$\hat{k} = \left(\frac{\beta^{-1}(1+g_q)^{\frac{1}{1-\alpha}} - (1-\delta)}{\hat{q}z\alpha}\right)^{\frac{1}{\alpha-1}} \tag{3}$$

Part (d)

The rental price of capital will be the marginal return of capital:

$$\begin{aligned} r_t &= z\alpha k_t^{\alpha-1} \\ &= z\alpha\left(\frac{1}{k_t}\right)^{1-\alpha} \end{aligned}$$

showing that the ratio of labor to capital is decreasing, and hence the rental rate of capital is decreasing. At the BGP we can write the rental rate explicitly as

$$\begin{aligned} r_t &= z\alpha\left(\frac{1}{k_0(1+g_q)^{\frac{t}{1-\alpha}}}\right)^{1-\alpha} \\ &= z\alpha k_0^{\alpha-1}(1+g_q)^{-t} \end{aligned} \tag{4}$$

Now at any equilibrium, we must have the return to saving equal to the return to investing. From (2) above, we have

$$\begin{aligned} \frac{u_{c,t}}{\beta u_{c,t+1}} &= i_t \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(\hat{q}z\alpha\hat{k}_{t+1}^{\alpha-1}+1-\delta\right) \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(\frac{q_t}{(1+g_q)^t}z\alpha\left(\frac{k_t}{(1+g_q)^{\frac{t}{1-\alpha}}}\right)^{\alpha-1}+1-\delta\right) \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(q_0(1+g_q)^t z\alpha k_0^{\alpha-1}(1+g_q)^{-t}+1-\delta\right) \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(q_0 z\alpha k_0^{\alpha-1}+1-\delta\right) \end{aligned} \tag{5}$$

showing that the interest rate is constant at the BGP. Basically: the return to capital *per unit* is falling, but because units of capital become cheaper over time relative to consumption, the return to investment (and hence the return to saving) is constant.

Part (e)

Repeating the steps in part (b), we arrive at the system of equations:

$$\begin{aligned} (1+g_e) &= (1+g)(1+g_q) \\ (1+g) &= (1+g_z)(1+g_e)^{\alpha_e}(1+g)^{\alpha_s} \end{aligned}$$

with solutions

$$1 + g_e = (1 + g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}$$
$$1 + g = (1 + g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}$$

Hence equipment grows at a faster rate than structures, and so the ratio of equipment to structures is growing over time. Now we wish to examine the rental rate and investment rate. Using the same notation as before, we have

$$\begin{aligned}
\hat{x}_t &= \frac{\left((1 + g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t+1} \hat{k}^e_{t+1} - (1 - \delta_e) \left((1 + g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^t \hat{k}^e_t}{\left((1 + g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^t (1 + g_q)^t \hat{q}_t} \\
&= \frac{\left((1 + g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t+1} \hat{k}^e_{t+1} - (1 - \delta_e) \left((1 + g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^t \hat{k}^e_t}{\left((1 + g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^t \hat{q}_t} \\
&= \frac{(1 + g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}} \hat{k}_{t+1} - (1 - \delta)\hat{k}_t}{\hat{q}} \\
\hat{i}_t &= (1 + g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}} \hat{k}^s_{t+1} + (1 - \delta_s)\hat{k}^s_t
\end{aligned}$$

And if we follow the same steps as in part (d), we obtain the rental rate

$$\begin{aligned}
r_t &= z_0(1 + g_z)^t \alpha \left(\frac{1}{k^e_0(1 + g_q)^{\frac{t-t\alpha_s}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{t}{1-\alpha_s-\alpha_e}}}\right)^{1-\alpha_e} \left(k^s_0(1 + g_q)^{\frac{t\alpha_e}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{t}{1-\alpha_s-\alpha_e}}\right)^{\alpha_s} \\
&= z_0\alpha_e(k^e_0)^{\alpha_e-1}(k^s_0)^{\alpha_s}(1 + g_q)^{-t(1-\alpha_s-\alpha_e)} \qquad (6)
\end{aligned}$$

which is again decreasing over time. Likewise the investment rate will be

$$\begin{aligned}
i_t &= \frac{\beta}{(1 + g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}} \left(\hat{q}\hat{z}_t\alpha_e(\hat{k}^e_t)^{\alpha_e-1}(\hat{k}^s_t)^{\alpha_s} + 1 - \delta\right) \\
&= \frac{\beta}{(1 + g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}} (1 + g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}} \left(q_0\alpha_e(k^e_0)^{\alpha_e-1}(k^s_0)^{\alpha_s} + 1 - \delta\right) \qquad (7)
\end{aligned}$$

which remains constant as before.

Problem 5

See the solutions manual to Ljungqvist and Sargent (it's easy to find).

Problem 6

Part (a)

For simplicity I assume that labor is inelastically supplied, with aggregate endowment equal to 1. With Pareto weights, we can write the problem as the following:

$$\max_{\{c_t^R, c_t^P, k_{t+1}^R, k_{t+1}^P\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\alpha_R \log(c_t^R - \bar{c}) + \alpha_P \log(c_t^P - \bar{c})\right) \ s.t.$$

$$c_t^R + c_t^P + k_{t+1}^R + k_{t+1}^P \leq A(k_t^R + k_t^P)^\alpha + (1-\delta)(k_t^R + k_t^P)$$

In any given period, the policymaker will face the static problem of dividing output between the rich and the poor. The solution to this intratemporal problem is given by the FOC's:

$$\begin{aligned}
\frac{\beta^t \alpha_R}{c_t^R - \bar{c}} &= \lambda_t \\
&= \frac{\beta^t \alpha_P}{c_t^P - \bar{c}} \\
\implies c_t^P &= \frac{\alpha_R - \alpha_P}{\alpha_R}\bar{c} + \frac{\alpha_P}{\alpha_R} c_t^R
\end{aligned}$$

implying that c_P is a convex combination of c_R and $\bar{c}$. Hence we can write the following:

$$\begin{aligned}
c_t &= \frac{c_t^R + (\alpha_R - \alpha_P)\bar{c}}{\alpha_R} \\
c_t^R &= \alpha_R c_t - (\alpha_R - \alpha_P)\bar{c} \\
c_t^P &= \alpha_P c_t + (\alpha_R - \alpha_P)\bar{c}
\end{aligned}$$

Hence we can write the problem recursively:

$$V(k) = \max_{k'} \left[\alpha_R \log(\alpha_R[Ak^\alpha - k'] + (1-\delta)k_t - 2\alpha_R\bar{c}) + \alpha_P \log(\alpha_P[Ak^\alpha + (1-\delta)k_t - k'] - 2\alpha_P\bar{c}) + \beta V(k')\right]$$

Part (b)

Taking FOC's,

$$\begin{aligned}
\frac{\alpha_R + \alpha_P}{Ak^\alpha + (1-\delta)k - k' - 2\bar{c}} &= \beta V'(k') \\
V'(k) &= \frac{(\alpha_R + \alpha_P)[A\alpha K^{\alpha-1} + 1 - \delta]}{Ak^\alpha + (1-\delta)k - k' - 2\bar{c}}
\end{aligned}$$

which gives us the Euler equation

$$\frac{1}{c_t - \bar{c}} = \beta \frac{A\alpha K^{\alpha-1} + 1 - \delta}{c_{t+1} - \bar{c}}$$

At the steady state, we will have $c_t = c_{t+1}$ and hence

$$k_{ss} = \left(\frac{1-\beta(1-\delta)}{\beta A\alpha}\right)^{\frac{1}{\alpha-1}}$$

which does not depend on the welfare weights.

Part (c)

From part (a), we have

$$c_t^R/c_t = \alpha_R - (\alpha_R - \alpha_P)\frac{\bar{c}}{c_t} \quad (1)$$

$$c_t^P/c_t = \alpha_P + (\alpha_R - \alpha_P)\frac{\bar{c}}{c_t} \quad (2)$$

Observe that as c_t becomes very large relative to $\bar{c}$, the right-hand sides of (1) and (2) drop out, and each group's consumption converges to their welfare weights. At the other extreme, where $c_t = 2\bar{c}$, each group consumes 1/2 of aggregate consumption. Hence, a high $\bar{c}$ tends to increase income equality, but it also means that inequality will change as aggregate consumption increases or decreases.

Assuming that $c_{ss} > 2\bar{c}$, in considering the evolution of inequality we need to consider two scenarios:

1. $c_0 > c_{ss}$: in this case, consumption becomes less unequal over time. Essentially, 'luxury' becomes less affordable over time, and consumption for both groups converges towards (though not necessarily to) subsistence levels.

2. $c_0 < c_{ss}$: now consumption becomes more unequal over time. Because growth in consumption is directed entirely in luxury or 'excess' consumption (i.e. consumption greater than $\bar{c}$, and because the wealthy are allowed to consume α_R of this excess consumption, the ratio c_t^R/c_t^p becomes greater over time. At the limit, this ratio converges to α_R/α_P.

And of course in the trivial case where $c_0 = c_{ss}$, inequality doesn't change at all.

Part (d)

Recalling that the only difference between the agents is their wealth/starting capital, consider the Euler equation for an individual of type i:

$$\frac{c_{t+1}-\bar{c}}{c_t-\bar{c}} = \beta(A\alpha k_{t+1}^{\alpha-1} + 1 - \delta)$$

$$\implies c_{t+1} = \bar{c} + (c_t - \bar{c})\beta(A\alpha k_{t+1}^{\alpha-1} + 1 - \delta)$$

Intuitively, agents can only save out of their 'excess' income, and so the savings rate of the rich will be higher than the savings rate of the poor, provided that $\bar{c} > 0$. If $\bar{c} = 0$, then savings (and also consumption) of both groups will grow at the same rate.

Let's assume that $k_0 < k_{ss}$, so that capital is converging upwards to the steady state. Then the term $\beta(A\alpha k_{t+1}^{\alpha-1} + 1 - \delta)$ is 'large' at $t = 0$, i.e. there is a high return to saving. If c_t is large relative to $\bar{c}$, then consumption grows quickly; if c_t is very close or equal to $\bar{c}$, then consumption grows more slowly, or perhaps doesn't grow at all. Hence, the initial wealthy will be able to capture more of the high return to investing, and it is precisely this which allows consumption inequality - and by the same token, wealth inequality k_t^R/k_t^P - to grow over time.

Note that the case where the poor's income is *just* greater than $\bar{c}$ corresponds to a case where $\alpha_p \to 0$: in this case, the rich will capture virtually all excess income. In other words, while initial capital levels determine the level of inequality, the evolution of inequality over time depends critically on $\bar{c}$.

Problem 7

For reference on this problem, see Phelps and Pollack (1968).

Part (a)

In this formulation, where the time 0 generation decides on all future consumption and investment, there is no inconsistency and we can solve the problem using the usual methods. Taking FOC:

$$c_0^{-1} = \lambda_0$$
$$\delta\beta^t c_t^{-1} = \lambda_t \ \forall\ t > 0$$
$$\lambda_t = R\lambda_{t+1}$$

Solving, we get

$$c_0 = \frac{c_1}{R\delta\beta}$$
$$c_t = \frac{c_{t+1}}{R\beta} \ \forall\ t > 0$$

Can we solve for c_t? Supposing that we had finite horizon T:

$$c_T = rk_T$$
$$c_{T-1} = \frac{rk_{T-1}}{1+\beta}$$
$$c_{T-2} = \frac{rk_{T-2}}{1+\beta+\beta^2}$$
$$...$$
$$c_t = (1-\beta)rk_t$$

We can solve for c_0 by noting that since $k' = \beta rk$,

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t \log([1-\beta]\beta^t r^t k_0) &= \sum_{t=0}^{\infty}\beta^t\left[t\log(\beta r) + \log(1-\beta) + \log k_0\right] \\ &= \sum_{t=0}^{\infty}\beta^t\left[t\log(\beta r)\right] + \frac{\log(1-\beta)}{1-\beta} + \frac{\log k_0}{1-\beta} \\ &= \frac{\beta}{(1-\beta)^2}\log(\beta r) + \frac{\log(1-\beta)}{1-\beta} + \frac{\log k_0}{1-\beta}\end{aligned}$$

and hence

$$c_0 = \frac{1-\beta}{1-\beta(1-\delta)} rk_0$$

Can we solve this using dynamic programming? Since the only 'special' period is the first period, and all subsequent periods are identical, we can divide the problem into two parts: the initial decision about k_1, and all future decisions. For periods $t > 0$, the problem can be solved recursively:

$$V(k; k_1) = \max_{k'}\left[\log(Rk - k') + \beta V(k'; k_1)\right]$$

and then the problem at time $t = 0$ becomes

$$U = \max_{k_1}\left[\log(Rk_0 - k_1) + \delta\beta V(k_1; k_1)\right] \tag{1}$$

Part (b)

With the addition of "t-selves", we introduce the problem of time-inconsistency. Now, the policy decision that maximizes the utility of the time 0 generation will come into conflict with the decision that maximizes the utility of any other generation. From the standpoint of 0-self, the optimal decision at time $t = 1$ is

$$\frac{1}{Rk_1 - k_2} = \beta V'(k_2; k_1) \tag{2}$$

whereas for 1-self, the optimal decision corresponds to (1) from above:

$$\frac{1}{Rk_1 - k_2} = \delta\beta V'(k_2; k_2) \tag{3}$$

The RHS of (3) is smaller than the RHS of (2), meaning that 1-self wants c_1 to be larger than would 0-self. Hence, if 0-self is allowed to dictate the savings policies for future generations, the utility of those generations will not be maximized.

Part (c)

The problem is now different: each generation takes future generations' savings decisions as given. The problem cannot be written as a Bellman because we still have hyperbolic discounting. However, we can write the problem as the choice of next-period capital k_{t+1}, given future savings behavior $g(k)$, that solves

$$\begin{aligned} &\max_{k_{t+1}} \; [\log(Rk_t - k_{t+1}) + \delta V(k_{t+1})] \qquad (4)\\ &V(k_{t+1}) = \sum_{s=t+1}^{\infty} \beta^{s-t} \log[Rk_s - g(k_s)] \\ &k_{t+1} \geq 0 \; \forall t \\ &k_t \text{ given} \end{aligned}$$

Part (d)

Suppose all future generations save at rate $g(k) = Ak$. Given this guess, the continuation value $V(k_{t+1})$ becomes

$$\begin{aligned} V(k_{t+1}) &= \sum_{s=t+1}^{\infty} \beta^{s-t} \log[(R-A)k_s] \\ &= \beta \left(\sum_{s=t+1}^{\infty} \beta^{s-t-1} \log[(R-A)A^{s-t-1}k_{t+1}] \right) \end{aligned}$$

Taking FOC of (4) w.r.t. k_{t+1},

$$\begin{aligned} \frac{1}{Rk_t - k_{t+1}} &= \delta\beta \sum_{s=t+1}^{\infty} \left(\beta^{s-t-1} \frac{(R-A)A^{s-t-1}}{(R-A)A^{s-t-1}k_{t+1}} \right) \\ &= \delta\beta \sum_{s=t+1}^{\infty} \left(\beta^{s-t-1} k_{t+1}^{-1} \right) \\ &= \frac{\delta\beta}{(1-\beta)k_{t+1}} \end{aligned}$$

Solving for k_{t+1}, we get

$$\begin{aligned} k_{t+1} &= \frac{\delta\beta}{1-\beta(1-\delta)} R k_t \\ &= A k_t \\ \implies c_t &= \frac{1-\beta}{1-\beta(1-\delta)} R k_t \end{aligned}$$

which verifies the guess. Note that t-self's decision is independent of the future savings rate A - this comes from the assumption of log utility.