Problem 1

Part (a)

We are given that the production function is Cobb-Douglas, and we are also given that labor supply is

$$\begin{aligned}\hat{N}_t &= N_t(\bar{e} - bn_t) \\ &= N_0 \prod_{i=0}^{t-1} n_i(\bar{e} - bn_t)\end{aligned}$$

Given this definition for $\hat{N}$, aggregate production will be

$$Y_t = K_t^{\alpha} \hat{N}_t^{1-\alpha}$$

and so we can write the feasibility constraint as

$$\begin{aligned}C_t + X_t &\leq K_t^{\alpha} \hat{N}_t^{1-\alpha} \\ K_{t+1} &= X_t + (1-\delta)K_t\end{aligned}$$

where $C_t, K_{t+1} \geq 0$ and $\delta \in [0, 1]$.

Part (b)

I assume for simplicity that the consumption and investment goods are identical, and normalize $p_t = 1$. Let $N_t^i = N_0^i \prod_{i=0}^{t-1} n_i$ where N_t^i is the initial size of dynasty i. On the income side of the dynasty's budget constraint, we will have wages of $N_{t-1}^i(\bar{e} - bn_t^i)w_t$ and rent of $k_t^i r_t$, as well as bond holdings from the previous period a_{t-1}. On the expense side we will have nominal consumption expenditure of c_t^i and investment x_t^i and bond purchases of $a_t q_t$. Hence we can write the budget constraint of an individual in dynasty i as

$$c_t^i + x_t^i + a_t q_t \leq w_t(\bar{e} - bn_t^i) + r_t k_t^i + a_{t-1}$$

and the aggregate budget constraint of all individuals in dynasty i as

$$C_t^i + X_t^i + N_t^i a_t q_t \leq w_t(\bar{e} - bn_t^i)N_t^i + r_t K_t^i + a_{t-1} N_t^i$$

Part (c)

If all households are identical, then we can write $c_t^i = c_t = C_t/N_t$. A CE is allocations for dynasties $\{c_t, x_t, k_t, n_t\}_{t=0}^{\infty}$ and the representative firm $\{Y_t^f, K_t^f, L_t^f\}_{t=0}^{\infty}$ and (relative) prices $\{p_t = 1, r_t, w_t\}_{t=0}^{\infty}$ such that

1. Dynasties solve the optimization problem

$$\max_{\{c_t, x_t, k_{t+1}, n_t\}_{t=0}^{\infty}} u(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_t) \prod_{i=0}^{t-1} g(n_i) \ s.t.$$

$$\sum_{t=0}^{\infty} [c_t + x_t] \leq \sum_{t=0}^{\infty} [w_t(\bar{e} - bn_t) + r_t k_t]$$

$$n_t k_{t+1} = x_t + (1-\delta)k_t$$

$$N_t = N_{t-1} n_{t-1}$$

$$c_t, n_t, k_{t+1} \geq 0$$

$$k_0 \text{ given}$$

2. The representative firm maximizes profits:

$$\max_{K_t^f, L_t^f} Y_t^f - r_t K_t^f - w_t L_t^f \ s.t.$$

$$Y_t^f = (K_t^f)^{\alpha} (L_t^f)^{1-\alpha}$$

3. Markets clear:

$$\text{Goods market}: \quad C_t + X_t = Y_t^f$$

$$\text{Capital market}: \quad K_t = K_t^f$$

$$\text{Labor market}: \quad (\bar{e} - bn_t) N_t = L_t^f$$

Part (d)

Making the usual assumptions about $u(\cdot)$, and putting everything in per capita terms, we can write the planner's problem as

$$\max_{c_t,k_{t+1},n_t} \quad u(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_t) \prod_{i=0}^{t-1} g(n_i) \tag{1}$$

$$\begin{aligned} s.t. \quad & c_t + n_t k_{t+1} = k_t^{\alpha}[(\bar{e} - bn_t)]^{1-\alpha} + (1-\delta)k_t \\ & N_{t+1} = N_t n_t \\ & c_t, k_{t+1} \geq 0 \\ & k_0 \text{ given} \end{aligned} \tag{2}$$

Now since the production function is homogeneous of degree one, we have

$$F(K_t, \hat{N}_t) = F_k(K_t, \hat{N}_t)K_t + F_n(K_t, \hat{N}_t)\hat{N}_t$$

and we know that at any competitive equilibrium, we will have

$$\begin{aligned} r_t K_t &= F_k(K_t, \hat{N}_t)K_t \\ w_t(\bar{e} - bn_t)N_t &= F_n(K_t, \hat{N}_t)\hat{N}_t \end{aligned} \tag{3}$$

Substituting this into the feasibility constraint and writing investment in terms of X, we have

$$c_t + x_t = w_t(\bar{e} - bn_t) + r_t k_t$$

which is the same as the dynasty's budget constraint. This implies, since all dynasties are the same, that the prices $\{p_t = 1, w_t, r_t\}_{t=0}^{\infty}$ that respect 3 ensure that any solution to the planner's problem is a competitive equilibrium.

Part (e)

Really this question is asking, what do we need to do to ensure that an interior solution exists? Joining the FOC w.r.t. n_0 and c_0 of (2):

$$\Big(- k_1 - bF_n(k_t, n_t)\Big) c_0^{-\sigma} + \sum_{t=1}^{\infty} \beta^t u(c_t) \left(\prod_{i=0}^{t-1} g(n_i) g'(n_0) \right) = 0$$

The first term is always negative, as is $u(c_t)$. Hence, for there to be any hope of an interior solution, we need $g'(\cdot)$ to be less than zero as well (note that $g(\cdot) < 0$ won't work). As Hakki points out, setting $g'(0) = -\infty$ ensures a positive number of children, and the solution will be interior since the left-most term goes to negative infinity as $bn_0 \to \bar{e}$ (because c_0 goes to 0), so that at some (unique) point in $(0, \bar{e}/b)$ the equality will hold.

Part (f)

First, we need to assume that $g(n_t) = g(n_{t'})\ \forall t, t'$ to be able to write the problem in recursive form. Note that in per capita terms, we need to adjusts next-period capital to reflect the increasing population. The recursive problem can be written as

$$V(k, N) = \max_{k' \in \Gamma(k), n \in [0, \bar{e}/b]} \Big(u\big[k^\alpha(\bar{e} - bn)^{1-\alpha} + (1-\delta)k - nk'\big] + g(n)\beta V(k', Nn) \Big)$$
$$\text{where } \Gamma(k) = \left[\frac{(1-\delta)k}{n}, \frac{k^\alpha(\bar{e} - nb)^{1-\alpha} + (1-\delta)k}{n} \right]$$

The lack of any role for the state variable N comes from the problem's assumption about the functional form of $g(\cdot)$ (i.e. that n enters as a divisor). Hence we can write this as

$$V(k) = \max_{k' \in \Gamma(k), n \in [0, \bar{e}/b]} \Big(u\big[k^\alpha(\bar{e} - bn)^{1-\alpha} + (1-\delta)k - nk'\big] + g(n)\beta V(k') \Big)$$

And only k is a state variable. Intuitively, n is not a state variable because our problem is formulated in *per capita terms*.

Part (g)

Substituting these assumptions into the objective function, we have

$$\frac{c_0^{1-\sigma}}{1-\sigma} + \sum_{t=1}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \prod_{i=0}^{t-1} n_i^{1+\eta} = \frac{c_0^{1-\sigma}}{1-\sigma} + \sum_{t=1}^{\infty} \beta^t N_t^{1+\eta} \frac{c_t^{1-\sigma}}{1-\sigma}$$

since $c_t = C_t/N_t$. Consider again the FOC with respect to n_0:

$$\Big(-k_1 - bF_n(k_t, n_t) \Big) c_0^{-\sigma} + \sum_{t=1}^{\infty} \beta^t N_t^{1+\eta} \frac{(1+\eta)}{n_0} \frac{c_t^{1-\sigma}}{1-\sigma} = 0$$

For there to be an interior solution, we require that

$$\frac{1+\eta}{1-\sigma} > 0$$

Therefore either $\sigma \in (0,1)$ and $\eta > -1$, or $\sigma > 1$ and $\eta < -1$. If these conditions are met then we will certainly have an interior solution, since the RH term 'blows up' as $n_0 \to 0$, and the LH term blows up as $n_0 \to \bar{e}/b$. We can also obtain these results by considering the FOC w.r.t. fertility for the recursive problem, which recall has only capital as a state variable:

$$c^{-\sigma}(bF_k + k') = (1+\eta)n^{\eta}\beta V(k')$$

The LHS is positive; $V(k)$ is negative if $\sigma > 1$, and positive if $\sigma < 1$, which in turn gives us the necessary conditions on η for an interior solution.

Part (h)

Writing the planner's problem as

$$\max_{\{k_{t+1},n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t^{\sigma+\eta} \frac{\left(K_t^{\alpha}(\bar{e}-bn)^{1-\alpha}N_t^{1-\alpha} + (1-\delta)K_t - K_{t+1}\right)^{1-\sigma}}{1-\sigma}$$

$$c_t, k_{t+1} \geq 0$$

$$k_0 \text{ given}$$

we want to first take FOC's

Capital :

$$C_t^{-\sigma} = \beta n_t^{\sigma+\eta} C_{t+1}^{-\sigma}\left(F_K(K_{t+1},\hat{N}_{t+1}) + (1-\delta)\right)$$

Fertility :

$$N_t^{\sigma+\eta} bN_t F_N(K_t,\hat{N}_t)C_t^{-\sigma} = \beta\left[(\sigma+\eta)N_t^{\sigma+\eta}n_t^{\sigma+\eta-1}\frac{C_{t+1}^{1-\sigma}}{1-\sigma} + N_t^{\sigma+\eta}n_t^{\sigma+\eta}F_N(K_{t+1},\hat{N}_{t+1})(\bar{e}-bn_{t+1})N_t C_{t+1}^{-\sigma}\right]$$

$$bN_t F_N(K_t,\hat{N}_t)C_t^{-\sigma} = \beta\left[(\sigma+\eta)n_t^{\sigma+\eta-1}\frac{C_{t+1}^{1-\sigma}}{1-\sigma} + n_t^{\sigma+\eta}F_N(K_{t+1},\hat{N}_{t+1})(\bar{e}-bn_{t+1})N_t C_{t+1}^{-\sigma}\right]$$

Where $(\bar{e}-bn_{t+1})N_t = \frac{d\hat{N}_{t+1}}{dn_t}$. Assuming $\bar{e}=1$

$$bN_t F_N(K_t,\hat{N}_t)C_t^{-\sigma} = \beta\left[(\sigma+\eta)n_t^{\sigma+\eta-1}\frac{C_{t+1}^{1-\sigma}}{1-\sigma} + n_t^{\sigma+\eta}F_N(K_{t+1},\hat{N}_{t+1})(1-bn_{t+1})N_t C_{t+1}^{-\sigma}\right]$$

Now at any BGP we will have $c_{t+1} = c_t$ and likewise for per capita capital and labor supply, which implies that aggregate variables grow at the same rate as population. Therefore, considering the Euler equation for capital, at any BGP we will have

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = n^{\sigma}$$

$$\implies 1 = \beta n^{\eta}(F_K + 1 - \delta)$$

noting that F_k will be constant given that K and N are growing at equal rates. Now considering the Euler equation for fertility,

$$\begin{aligned} bN_tF_N(K_t,\hat{N}_t)C_{t+1}^{-\sigma}n^{\sigma} &= \beta\left[(\sigma+\eta)n_t^{\sigma+\eta-1}\frac{C_{t+1}^{1-\sigma}}{1-\sigma} + n_t^{\sigma+\eta}F_N(K_{t+1},\hat{N}_{t+1})(1-bn_{t+1})N_tC_{t+1}^{-\sigma}\right] \\ bF_N(K_t,\hat{N}_t) &= \beta\left[(\sigma+\eta)n_t^{\eta-1}\frac{1}{N_t}\frac{C_{t+1}}{1-\sigma} + n_t^{\eta}F_N(K_{t+1},\hat{N}_{t+1})(1-bn_{t+1})\right] \\ bF_N(K_t,\hat{N}_t) &= \beta\left[(\sigma+\eta)n_t^{\eta-1}\frac{n_t}{N_{t+1}}\frac{C_{t+1}}{1-\sigma} + n_t^{\eta}F_N(K_{t+1},\hat{N}_{t+1})(1-bn_{t+1})\right] \\ \implies bF_n &= \beta n^{\eta}\left[\frac{\sigma+\eta}{1-\sigma}c + F_n(1-bn)\right] \text{ on the BGP} \end{aligned}$$

Notice that $f_t = F_{Nt} = F_{Nt'}$ $\forall t, t'$ on the BGP since the derivative of an HD1 function is HD0. Since both arguments of F grow at a constant rate, then the derivative at t is equal to the derivative at $t+1$. Together with the budget constraint, these Euler equations characterize the BGP. They comprise a system of equations in three unknowns, c, n, k and its solution pins down the long-run growth in a BGP.

Part (i)

Note the typo: $-\sigma = \eta$ instead of $1-\sigma = \eta$. With this substitution the Euler equation for fertility becomes

$$b = \beta(n^{\eta} - bn)$$

Rewriting $\hat{N}_t = (\bar{e} - bn)H_t$ and writing the law of motion for H_t as

$$H_{t+1} = nH_t$$

we can view population increase as investment in human capital (without depreciation). This is similar to the AKH model of growth.

Part (j)

Recall that at the BGP, per capita variables are constant so *all* growth in the economy is coming through population growth. In reality, we see growth in per capita variables as well. This implies that productivity is improving over time - a dynamic missing from this model. Hakki makes some good points on his solutions.

Problem 2

Part (1)

A competitive equilibrium is allocations $\{c_t^i, k_t^i, n_t^i, x_t^i\}_{t=0}^{\infty}$, $\{S_t, O_t, K_t, N_t\}_{t=0}^{\infty}$ and prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$ such that

Households solve the problem

$$\begin{aligned}
\max_{\{c_t^i, k_t^i\}} & \int_{t=0}^{\infty} e^{-\rho t} \ln c_t^i dt \ s.t. \\
& \int_{t=0}^{\infty} (p_t(c_t^i + x_t^i) - r_t k_t^i - w_t n_t^i) dt \leq 0 \\
& \dot{k} = -\delta k + x_t^i \\
& c_t^i, k_t^i, n_t^i \geq 0 \\
& n_k^i \leq e \\
& k_0 \text{ given}
\end{aligned}$$

Firms maximize:

$$\begin{aligned}
\max_{\{S_t, O_t, K_t, N_t\}} & p_t(S_t^{-\sigma} O_t^{\alpha} K_t^{\beta} N_t^{1-\alpha-\beta}) - r_t K_t - w_t N_t \ s.t. \\
& \dot{S}_t = -\gamma S_t + O_t \\
& \dot{R}_t = -O_t \\
& R_0 \text{ given}
\end{aligned}$$

Markets clear:

$$\begin{aligned}
K_t &= \int_i k_t^i di \\
N_t &= \int_i n_t^i di \\
C_t + \dot{K}_t + \delta K_t &= Y_t
\end{aligned}$$

Part (2)

Planner's problem:

$$\max_{\{C_t,S_t,O_t,K_tN_t\}} \int_{t=0}^{\infty} e^{-\rho t}\ln C_t dt \ \ s.t.$$

$$C_t + \dot{K}_t + \delta K_t = S_t^{-\sigma}O_t^{\alpha}K_t^{\beta}N_t^{1-\alpha-\beta}$$

$$\dot{S}_t = -\gamma S_t + O_t$$

$$\dot{R}_t = -O_t$$

$$S_0, K_0, R_0 \ \text{ given}$$

Part (3)

The Lagrangian of the planner's problem is as follows

$$L = \int_{t=0}^{T} e^{-\rho t}\ln C_t dt + \int_{t=0}^{T} \lambda_t[S_t^{-\sigma}O_t^{\alpha}K_t^{\beta}N_t^{1-\alpha-\beta} - C_t - \dot{K}_t - \delta K_t]dt$$
$$+ \int_{t=0}^{T} \mu_t[\dot{S}_t + \gamma S_t - O_t]dt + \nu[R_0 - \int_{t=0}^{T} O_t dt]$$

Using integration by parts:

$$\int_{t=0}^{T} \lambda_t \dot{K}_t dt = \lambda_T K_T - \lambda_0 K_0 - \int_{t=0}^{T} \dot{\lambda}_t K_t dt$$
$$\int_{t=0}^{T} \mu_t \dot{S}_t dt = \mu_T S_T - \mu_0 S_0 - \int_{t=0}^{T} \dot{\mu}_t S_t dt$$

Therefore we get

$$L = \int_{t=0}^{T} e^{-\rho t}\ln C_t dt + \int_{t=0}^{T} [\lambda_t(S_t^{-\sigma}O_t^{\alpha}K_t^{\beta}N_t^{1-\alpha-\beta} - C_t - \delta K_t) + \dot{\lambda}_t K_t]dt - \lambda_T K_T + \lambda_0 K_0$$
$$+ \int_{t=0}^{T} [\mu_t(\gamma S_t - O_t) - \dot{\mu}_t S_t]dt + \mu_T S_T - \mu_0 S_0 + \nu[R_0 - \int_{t=0}^{T} O_t dt]$$

F.O.C's: (note that I have only written the final result of combining FOC's for C_t and K_t. The procedure to get there is identical to the one in the lecture notes).

$$[C_t, K_t] : \frac{\dot{C}_t}{C_t} = -\frac{\dot{\lambda}}{\lambda} - \rho = f'(K_t) - \delta - \rho \tag{4}$$

$$S_t : \lambda_t f'(S_t) + \mu_t \gamma - \dot{\mu}_t = 0 \tag{5}$$

$$O_t : \lambda_t f'(O_t) - \mu_t - \nu = 0 \tag{6}$$

First, from the law of motion of S_t we have

$$\frac{\dot{S}_t}{S_t} = -\gamma + \frac{O_t}{S_t}$$

Therefore, on the BGP $\frac{O_t}{S_t}$ is a constant because the growth rate of S is constant, which implies O_t and S_t are growing at the same rate. Next, note that $f'(K_t)$ needs also to be constant [because of (4)]

$$f'(K_t) = \beta S_t^{-\sigma} O_t^{\alpha} K_t^{\beta-1} N_t^{1-\alpha-\beta} = S_t^{\alpha-\sigma} \left(\frac{O_t}{S_t}\right)^{\alpha} K_t^{\beta-1} N_t^{1-\alpha-\beta}$$

Since $\frac{O_t}{S_t}$ is constant and we don't have population growth so N_t is constant too, then it must be that $S_t^{\alpha-\sigma} K_t^{\beta-1}$ is constant as well.

$$\begin{aligned}
&\frac{d}{dt} S_t^{\alpha-\sigma} K_t^{\beta-1} = S_t^{\alpha-\sigma-1} K_t^{\beta-2}[(\alpha-\sigma)K_t\dot{S}_t + (\beta-1)S_t\dot{K}_t] \\
&= S_t^{\alpha-\sigma} K_t^{\beta-1}[(\alpha-\sigma)\frac{\dot{S}_t}{S_t} + (\beta-1)\frac{\dot{K}_t}{K_t}] = S_t^{\alpha-\sigma} K_t^{\beta-1}[(\alpha-\sigma)w + (\beta-1)]\frac{\dot{K}_t}{K_t} = 0 \\
&\iff w = \frac{1-\beta}{\alpha-\sigma}
\end{aligned}$$

Where I have defined w such that $\frac{\dot{S}_t}{S_t} = w\frac{\dot{K}_t}{K_t}$. Next, from F.O.C's we have

$$\begin{aligned}
\lambda_t f'(S_t) &= -\mu_t\gamma + \dot{\mu}_t \\
\lambda_t f'(O_t) &= \mu_t + \nu \\
\implies \lambda_t f'(S_t) &= \dot{\mu}_t - \gamma\lambda_t f'(O_t) + \gamma\nu \\
f'(S_t) &= \frac{\dot{\mu}_t}{\lambda_t} - \gamma f'(O_t) + \gamma\frac{\nu}{\lambda_t} \qquad (7)
\end{aligned}$$

Note that the last term goes to zero. Because the denominator is growing constantly by time (by (4)), while ν stays constant. Even without realizing this, you can take the second derivative ($\ddot{\mu}$) and work with $\dot{\mu}$ and $\ddot{\mu}$, so that you don't have to deal with ν. From (6) we can take the derivative with respect to time and write $\dot{\mu}_t$ as

$$\dot{\mu}_t = \dot{\lambda}_t f'(O_t) + \lambda_t \nabla f'(O_t). \begin{bmatrix} \dot{S}_t \\ \dot{O}_t \\ \dot{K}_t \end{bmatrix}$$

$$\implies \text{plug into (7) } f'(S_t) = \frac{\dot{\lambda}_t}{\lambda_t} f'(O_t) + \nabla f'(O_t). \begin{bmatrix} \dot{S}_t \\ \dot{O}_t \\ \dot{K}_t \end{bmatrix} - \gamma f'(O_t)$$

We divide both sides by $f'(O_t)$

$$\frac{f'(S_t)}{f'(O_t)} = \frac{\dot{\lambda}_t}{\lambda_t} + \frac{1}{f'(O_t)} \left(\nabla f'(O_t). \begin{bmatrix} \dot{S}_t \\ \dot{O}_t \\ \dot{K}_t \end{bmatrix} \right) - \gamma \qquad (8)$$

Now note that $f'(S_t) = -\sigma \frac{Y_t}{S_t}$ and $f'(O_t) = \alpha \frac{Y_t}{O_t}$, Therefore we have $\frac{f'(S_t)}{f'(O_t)} = -\frac{\sigma}{\alpha}\frac{O_t}{S_t}$ (constant). Next, observe that we have

$$\nabla f'(O_t). \begin{bmatrix} \dot{S}_t \\ \dot{O}_t \\ \dot{K}_t \end{bmatrix} = \frac{\partial Y'(O_t)}{\partial S_t}\dot{S}_t + \frac{\partial Y'(O_t)}{\partial O_t}\dot{O}_t + \frac{\partial Y'(O_t)}{\partial K_t}\dot{K}_t$$

$$= -\alpha\sigma \frac{Y_t}{O_t}\frac{\dot{S}_t}{S_t} + \alpha \frac{Y'(O_t)O_t - Y_t}{O_t}\frac{\dot{O}_t}{O_t} + \alpha\beta \frac{Y_t}{O_t}\frac{\dot{K}_t}{K_t}$$

$$\implies \frac{1}{Y'(O_t)}\nabla f'(O_t). \begin{bmatrix} \dot{S}_t \\ \dot{O}_t \\ \dot{K}_t \end{bmatrix} = -\sigma \frac{\dot{S}_t}{S_t} + (\alpha - 1)\frac{\dot{O}_t}{O_t} + \beta \frac{\dot{K}_t}{K_t} = ((-\sigma + \alpha - 1)w + \beta)\frac{\dot{K}_t}{K_t} \tag{9}$$

Now going back to (8) and replacing $\frac{f'(S_t)}{f'(O_t)}$ and (9) we get

$$-\frac{\sigma}{\alpha}\frac{O_t}{S_t} = \frac{\dot{\lambda}_t}{\lambda_t} + ((-\sigma + \alpha - 1)w + \beta)\frac{\dot{K}_t}{K_t} - \gamma$$

$$-\sigma(wx + \gamma) = \alpha(-x - \rho) + \alpha\left((-\sigma + \alpha - 1)w + \beta\right)x - \alpha\gamma$$

$$[\alpha(-1 + \beta + w(-\sigma + \alpha - 1)) + \sigma w]x = -\sigma\gamma + \alpha(\rho + \gamma)$$

$$x = \frac{\sigma\gamma - \alpha(\gamma + \rho)}{1 - \beta}$$

Where $x = \frac{\dot{K}_t}{K_t}$.

Part (4)

In order for the competitive equilibrium to achieve efficiency we need firms to consume oil and emit carbon at the efficient level, and price their good accordingly. This can be achieved if they have to pay a cost for oil and emissions that reflects their dynamic impact on social welfare. In other words, we need to have firms (which by definition solve a static problem period by period) behave as if they were forward looking. In fact, in our setting, externalities are between the same firm in different points in time, rather than across different agents in the model. The textbook policy is to implement a Pigouvian tax that prices emissions and oil at their marginal social cost.

Problem 3

Part (a)

Question: Define an equilibrium and characterize the market clearing factor prices and determine the free-entry condition.

An equilibrium in this economy is allocations for the household that consist of consumption levels, machine expenditures, and research expenses $[C(t), X(t), Z(t)]_{t=0}^{\infty}$; allocations for the machine-producing (intermediate good-producing) firms $[p^x(\nu,t), x(\nu,t)]_{t=0}^{\infty}$; prices $[r(t), w(t)]_{t=0}^{\infty}$; and time path of aggregate technology $[N(t)]_{t=0}^{\infty}$, such that 1) households maximize utility, 2) firms maximize discounted value of profits[1], 3) markets clear, and 4) aggregate technology is consistent with free entry and determined by by the LoM

$$\dot{N}(t) = \eta N(t)^{-\phi} Z(t)$$

Monopolists' prices:

Start with the consumption good producer's problem:

$$\max_{\{x(v,t)\}_{v\in[0,N(t)]},\, L} \frac{1}{1-\beta}\left(\int_0^{N(t)} x(v,t)^{1-\beta} dv\right) L^{\beta} - \int_0^{N(t)} p^x(v,t)x(v,t)dv - w(t)L$$

FOCs:

$$x(v,t): \quad x(v,t)^{-\beta}L^{\beta} = p^x(v,t) \implies \text{demand } x(v,t) = p^x(v,t)^{-\frac{1}{\beta}}L$$

$$L: \quad \frac{\beta}{1-\beta}\left(\int_0^{N(t)} x(v,t)^{1-\beta} dv\right) L^{\beta-1} = w(t)$$

Then the intermediate goods producers, who at time t maximize their present value knowing the demand for their output:

$$\max_{p^x(v,t)} \int_t^{\infty} exp\left(-\int_t^s r(s')ds'\right) \pi(v,s)ds$$

where $\pi(v,s) \equiv (p^x(v,t) - \psi)x(v,t)$ where $\psi > 0$ is the marginal cost of producing one unit of intermediate good. Since the demand for intermediate products does not depend on $r(t)$, the

[1] Unlike other models we have seen, in which the number of firms is fixed, here we have an endogenously determined number of firms. This number is pinned down by a free entry condition that equates the sunk cost of entry with its return: the present value of the infinite flow of profits.

problem is equivalent to maximizing profits at each period:

$$\max_{p^x(v,t)} (p^x(v,t) - \psi)x(v,t)$$
$$= \max_{p^x(v,t)} (p^x(v,t) - \psi)p^x(v,t)^{-\frac{1}{\beta}} L$$

FOCs:

$$p^x(v,t): \quad p^x(v,t)^{\frac{-1}{\beta}} L = \frac{1}{\beta}(p^x(v,t) - \psi)p^x(v,t)^{-\frac{1-\beta}{\beta}} L$$

Simplifying yields $p^x(v,t) = \frac{\psi}{1-\beta} \implies x(v,t) = L$ and to simplify notation, assume $\psi = 1-\beta \implies p^x(v,t) = 1 \; \forall v,t$

From the consumption good producer's FOC w.r.t L:

$$w(t) = \frac{\beta}{1-\beta}\left(\int_0^{N(t)} x(v,t)^{1-\beta} dv\right) L^{\beta-1}$$
$$= \frac{\beta}{1-\beta}\left(\int_0^{N(t)} L^{1-\beta} dv\right) L^{\beta-1}$$
$$= \frac{\beta}{1-\beta} N(t)$$

From the HH problem the Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t) - \rho) \implies r(t) = \theta\frac{\dot{C}(t)}{C(t)} + \rho$$

The free entry condition:

The innovation possibilities frontier $\dot{N}(t) = \eta N(t)^{-\phi} Z(t)$ means that given every unit of consumption good spent to create a new variety of intermediate good will turn into $\eta N(t)^{-\phi}$ new varieties, and therefore yields $\eta N(t)^{-phi}$ times the discounted sum of the respective firm's profits. The free entry condition imposes that the value of investing in creating a new variety is less or equal to the sunk cost of one unit of consumption good.

$$\eta N(t)^{-\phi} V(v,t) \leq 1$$
$$Z(t) \geq 0$$
$$(\eta N(t)^{-\phi} V(v,t)) Z(t) = 0$$

Part (b)-(c)

We are going to prove that on the BGP the real interest rate is constant, hence the NPV of an intermediate good firm grows as fast as labor does, which implies that the number of varieties grows as fast as labor, which finally implies that output grows as fast as labor. Start with the consumer's Euler and the fact that on the BGP the growth rate of consumption is constant. I am going to use * to denote variables on the BGP:

$$r^*(t) = \theta \frac{\dot{C}^*(t)}{C^*(t)} + \rho = \theta g_C + \rho = r^*$$

Plug in the NPV:

$$\begin{aligned} V^*(v,t) &= \int_t^\infty exp\left(-\int_t^s r^*(s')ds'\right)\pi(v,s)ds \\ &= \int_t^\infty exp\left(-\int_t^s r^* ds'\right)(1-\psi)L^* ds \\ &= \int_t^\infty e^{-r^*(s-t)}(1-\psi)L^* ds \\ &= (1-\psi)L^* e^{r^*t}\int_t^\infty e^{-r^*s}ds \\ &= (1-\psi)L^* e^{r^*t}\left[-\frac{1}{r^*}e^{-r^*s}\right]_{s=t}^\infty \\ &= \frac{(1-\psi)L^*}{r^*} \\ &= \frac{\beta L^*}{r^*} \end{aligned}$$

And hence looking at time derivatives:

$$\dot{V}^*(v,t) = \frac{(1-\psi)\dot{L}^*}{r^*} \implies \frac{\dot{V}^*(v,t)}{V^*(v,t)} = \frac{\dot{L}^*}{L^*} \equiv g_L$$

Now from the free-entry condition (binding):

$$\begin{aligned} N^*(t) &= (\eta V^*(v,t))^{\frac{1}{\phi}} = \left(\frac{\eta\beta L^*}{r^*}\right)^{\frac{1}{\phi}} \\ \implies \dot{N}(t) &= \frac{1}{\phi}\left(\frac{\eta\beta L^*}{r^*}\right)^{\frac{1}{\phi}-1}\frac{\eta\beta\dot{L}^*}{r^*} \\ &= \frac{1}{\phi}N(t)\frac{\dot{L}^*}{L^*} \\ \implies \frac{\dot{N}(t)}{N(t)} &= \frac{1}{\phi}\frac{\dot{L}^*}{L^*} = \frac{1}{\phi}g_L \end{aligned}$$

Now, look at the consumption good production function:

$$\begin{aligned} Y^*(t) &= \frac{1}{1-\beta}\left(\int_0^{N(t)} x(v,t)^{1-\beta}dv\right)L^{*\beta} = \frac{1}{1-\beta}N^*(t)L^* \\ \implies \frac{\dot{Y}^*(t)}{Y(t)} &= \frac{\dot{N}^*(t)}{N^*(t)} + \frac{\dot{L}^*}{L^*} = \left(1+\frac{1}{\phi}\right)g_L \end{aligned}$$

Hence if $g_L = 0$ then $g_Y = 0$ too.

Part (c)

Given $g_N = \frac{1}{\phi} g_L$, $g_Y = (1 + 1/\phi) g_L$, and $g_L = n$ then $g_Y = (1 + 1/\phi)n$

Problem 4

Part (a)

I assume a representative agent for each country. In autarky, a competitive equilibrium for country j is consumption C_j and allocations for intermediate firms $[x_j(\omega), k_j(\omega)]_{\omega\in[0,1]}$, along with prices $[p_j(\omega), r_j]_{\omega\in[0,1]}$ (normalizing the price of the final good to one in each country), such that

1. Households maximize utility:

$$\max_{C_j} u(C_j) \ s.t.$$
$$C_j \leq r_j K_j$$
$$K_j \text{ given}$$

2. Intermediate firms maximize profits:

$$\max_{k_j(\omega)} p_j(\omega) A_j(\omega) k_j(\omega) - r_j k_j(\omega)$$

3. The final firm maximizes profits:

$$\max_{x_j(\omega)} e^{\int_0^1 \log x_j(\omega) d\omega} - \int_0^1 p_j(x_j(\omega)) x_j(\omega) d\omega$$

4. Markets clear:

$$\log C_j = \log Y_j = \int_0^1 \log x_j(\omega) d\omega$$
$$x_j(\omega) = A_j(\omega) k_j(\omega)$$
$$K_j = \int_0^1 k_j(\omega) d\omega$$

Part (b)

Taking the FOC for an intermediate firm, we have

$$A_j(\omega) p_j(\omega) = r_j$$

Likewise, for the final firm we have

$$x_j(\omega)p_j(\omega) = Y_j$$

From the market-clearing condition for intermediate goods, it follows that

$$\begin{aligned} Y_j &= A_j(\omega)k_j(\omega)p_j(\omega) \\ &= r_j k_j(\omega) \\ &= r_j k_j \end{aligned}$$

indicating that an equal amount of capital is rented by all intermediate goods-producers, and that therefore given the assumption of a unit continuum of producers and market clearing,

$$\begin{aligned} K_j &= \int_0^1 k_j d\omega = k_j \int_0^1 d\omega = k_j \\ \implies x_j(\omega) &= A_j(\omega)K_j \end{aligned}$$

The production function for the final good therefore becomes

$$Y_j = e^{\int_0^1 \log(A_j(\omega)K_j)d\omega}$$

Noting that the functional form assumptions for $A_j(\omega)$ imply that

$$\int_0^1 \log(A_1(\omega)K_1)d\omega = \int_0^1 \log(z_1\omega K_1)d\omega = \log(z_1K_1) + [x\log x - 1]_0^1 \approx \log(z_1K_1) - 1$$
$$\int_0^1 \log(A_2(\omega)K_2)d\omega = \int_0^1 \log(z_2[1-\omega]K_2)d\omega = \log(z_2K_2) + [x\log x - 1]_0^1 \approx \log(z_2K_2) - 1$$

we have

$$Y_1 = z_1K_1e^{-1}$$
$$Y_2 = z_2K_2e^{-1}$$

Finally, we can solve for prices and the rental rate of capital:

$$\begin{aligned} r_j &= Y_j/K_j \\ &= \begin{cases} z_1e^{-1} & i=1 \\ z_2e^{-1} & i=2 \end{cases} \\ p_j(\omega) &= Y_j/K_jA_j(\omega) \\ &\approx \begin{cases} \frac{1}{e\omega} & i=1 \\ \frac{1}{e(1-\omega)} & i=2 \end{cases} \end{aligned}$$

Note that the endpoints $\{0, 1\}$ are problematic these last few steps; we really need to add the assumption that in country one, there is no producer for $\omega = 0$, whereas in country 2 there is no producer for $\omega = 1$.

Part (c)

With trade in intermediate goods, it is evident that both final goods-producing firms will use the same proportions of inputs. Hence a CE is the same as above, except that now the problem for final goods firm i is

$$\max_{x_1^i(\omega), x_2^i(\omega)} e^{\int_0^1 \log[x_1^i(\omega)+x_2^i(\omega)]d\omega} - \int_0^1 [p_1(x_1^i(\omega))x_1^i(\omega) + p_2(x_2^i(\omega))x_2^i(\omega)]d\omega$$

and market-clearing for intermediate goods is now

$$x_i^1(\omega) + x_i^2(\omega) = A_i(\omega)k_i(\omega)$$

Additionally, the prices needed for market clearing are now $[p(\omega), p_1, p_2]_{\omega=0}^{\infty}$. Trade implies that prices for intermediate goods will equalize, and we also need to take into account that prices of final goods may be different between the two countries (although in fact, they won't be).

Part (d)

By assumption, $A_1(0) = 0$ and $A_1'(\omega) = z_1$, whereas $A_2(1) = 0$ and $A_2'(\omega) = 1 - \omega$. Hence, considering the term

$$\frac{A_1(\omega)}{A_2(\omega)} = \frac{z_1\omega}{z_2(1-\omega)}$$

we know that this term is 0 when $\omega = 0$, is strictly increasing in ω, and goes to $+\infty$ at the limit. Now since the FOC for intermediate-good producers is $p_j(\omega) = r_j/A_j(\omega)$, we have

$$\begin{aligned}\frac{p_1(\omega)}{p_2(\omega)} &= \frac{r_1}{r_2} \times \frac{A_2(\omega)}{A_1(\omega)} \\ &= \frac{r_1}{r_2} \times \frac{z_2(1-\omega)}{z_1\omega}\end{aligned}$$

From the previous result, given any $r_1, r_2 \in \mathbb{R}^2_{++}$, there will exist a unique ω^* such that

$$\begin{aligned}r_1 A_2(\omega^*) &= r_2 A_1(\omega^*) \\ r_1 z_2 \omega^* &= r_2 z_1 (1 - \omega^*) \\ (r_1 z_2 + r_2 z_1)\omega^* &= r_2 z_1 \\ \omega^* &= \frac{r_2 z_1}{r_1 z_2 + r_2 z_1}\end{aligned}$$

and we will have $p_1(\omega') < p_2(\omega')$ for $\omega' > \omega^*$, and $p_1(\omega'') > p_2(\omega'')$ for $\omega'' < \omega^*$. Since $p_1(\omega)$ and $p_2(\omega)$ are perfect substitutes for any ω, it is evident that the firm will only purchase the lower-priced alternative, which in turn implies that it will not be profitable for intermediate firms in country 1 to produce for $\omega < \omega^*$, whereas intermediate firms in country 2 will not produce for $\omega > \omega^*$.

The FOC for firms are basically unchanged from above:

$$A_j(\omega)p(\omega) = r_j$$
$$x_i(\omega)p(\omega)/p_i = Y_i$$

and substituting these into the market-clearing condition with trade,

$$Y_i = \begin{cases} r_1(\omega)k_1(\omega)/p_i & \omega \geq \omega^* \\ r_2(\omega)k_2(\omega)/p_i & \omega < \omega^* \end{cases}$$

Hence, in this case demand for intermediate goods is given by

$$x_1(\omega) + x_2(\omega) = \begin{cases} \frac{A_1(\omega)K_1}{1-\omega^*} & \omega \geq \omega^* \\ \frac{A_2(\omega)K_2}{\omega^*} & \omega < \omega^* \end{cases}$$

Now note that the production function for final goods is homogeneous of degree 1, so that the only difference between the two firms is that the producer for country 1 will be allocated some portion α of both inputs and production, and the other firm will receive and produce a portion $1-\alpha$. But then the firms face the same marginal cost, implying that in this economy, equalization of intermediate goods prices implies equalization of final goods prices, which we can normalize to 1. Final production for country i is therefore given by

$$\begin{aligned}
\log(Y_1 + Y_2) &= \int_{\omega^*}^{1} \log\left[A_1(\omega)\frac{K_1}{1-\omega^*}\right] d\omega + \int_0^{\omega^*} \log\left[A_2(\omega)\frac{K_2}{\omega^*}\right] d\omega \\
&= \left[x \log\left(\frac{z_1K_1}{1-\omega^*}x\right) - x\right]_{\omega^*}^{1} + \left[x \log\left(\frac{z_2K_2}{\omega^*}x\right) - x\right]_{1-\omega^*}^{1} \\
&= \log\left(\frac{z_1K_1z_2K_2}{\omega^*(1-\omega^*)}\right) - \omega^* \log\left(\frac{\omega^*}{1-\omega^*}z_1K_1\right) - (1-\omega^*)\log\left(\frac{1-\omega^*}{\omega^*}z_2K_2\right) - 1
\end{aligned}$$

At this point we can solve for prices,

$$\begin{aligned}
p(\omega) &= \frac{Y}{x(\omega)} \\
&= \begin{cases} \frac{(1-\omega^*)z_1K_1z_2K_2(\omega^*[1-\omega^*])^{-1}\left(\frac{\omega^*}{1-\omega^*}z_1K_1\right)^{-\omega^*}\left(\frac{1-\omega^*}{\omega^*}z_2K_2\right)^{\omega^*-1}e^{-1}}{\omega z_1K_1} & \omega \geq \omega^* \\ \frac{w^*z_1K_1z_2K_2(\omega^*[1-\omega^*])^{-1}\left(\frac{\omega^*}{1-\omega^*}z_1K_1\right)^{-\omega^*}\left(\frac{1-\omega^*}{\omega^*}z_2K_2\right)^{\omega^*-1}e^{-1}}{(1-\omega)z_2K_2} & \omega < \omega^* \end{cases} \\
&= \begin{cases} (\omega^*)^{-1}\left(z_1K_1\right)^{-\omega^*}\left(z_2K_2\right)^{\omega^*}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}\frac{1}{\omega} & \omega \geq \omega^* \\ (1-\omega^*)^{-1}(z_1K_1)^{1-\omega^*}(z_2K_2)^{\omega^*-1}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}\frac{1}{1-\omega} & \omega < \omega^* \end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
r_1 &= z_1(\omega^*)^{-1}\left(z_1K_1\right)^{-\omega^*}\left(z_2K_2\right)^{\omega^*}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1} \\
r_2 &= z_2(1-\omega^*)^{-1}(z_1K_1)^{1-\omega^*}(z_2K_2)^{\omega^*-1}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}
\end{aligned}$$

This tells us, first of all, the value of ω^*, since from the beginning of this section

$$\begin{aligned}
r_1A_2(\omega^*) &= r_2A_1(\omega^*) \\
\implies (1-\omega^*)^2z_2K_2 &= (\omega^*)^2z_1K_1 \\
\implies \omega^* &= \frac{(K_2z_2)^{1/2}}{(K_1z_1)^{1/2}+(K_2z_2)^{1/2}}
\end{aligned}$$

But then we have also solved for relative rents, since

$$\begin{aligned}
\frac{r_1}{r_2} &= \frac{z_1(\omega^*)^{-1}\left(z_1K_1\right)^{-\omega^*}\left(z_2K_2\right)^{\omega^*}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}}{z_2(1-\omega^*)^{-1}(z_1K_1)^{1-\omega^*}(z_2K_2)^{\omega^*-1}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}} \\
&= \frac{K_2(1-\omega^*)}{K_1\omega^*} \\
&= \frac{(K_2z_1)^{1/2}}{(K_1z_2)^{1/2}}
\end{aligned}$$

Using these results we can solve for production:

$$\begin{aligned}
Y_1+Y_2 &= z_1K_1z_2K_2(\omega^*[1-\omega^*])^{-1}\left(\frac{\omega^*}{1-\omega^*}z_1K_1\right)^{-\omega^*}\left(\frac{1-\omega^*}{\omega^*}z_2K_2\right)^{\omega^*-1}e^{-1} \\
&= z_1K_1z_2K_2\frac{\left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right]^2}{(z_1K_1z_2K_2)^{1/2}}\left(\left(\frac{K_2z_2}{K_1z_1}\right)^{1/2}z_1K_1\right)^{-\omega^*}\left(\left(\frac{K_1z_1}{K_2z_2}\right)^{1/2}z_2K_2\right)^{\omega^*-1}e^{-1} \\
&= \left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right]^2e^{-1}
\end{aligned}$$

We can now solve for quantities and prices:

$$p(\omega) = \begin{cases} \frac{\left[(K_1 z_1)^{1/2} + (K_2 z_2)^{1/2}\right]}{(K_1 z_1)^{1/2}} e^{-1} \frac{1}{\omega} & \omega \geq \omega^* \\ \frac{\left[(K_2 z_2)^{1/2} + (K_2 z_2)^{1/2}\right]}{(K_2 z_2)^{1/2}} e^{-1} \frac{1}{1-\omega} & \omega < \omega^* \end{cases}$$

$$r_1 = z_1^{1/2} \frac{\left[(K_1 z_1)^{1/2} + (K_2 z_2)^{1/2}\right]}{K_1^{1/2}} e^{-1}$$

$$r_2 = z_2^{1/2} \frac{\left[(K_1 z_1)^{1/2} + (K_2 z_2)^{1/2}\right]}{K_2^{1/2}} e^{-1}$$

$$x(\omega) = \begin{cases} (z_1 K_1)^{1/2} \left[(K_1 z_1)^{1/2} + (K_2 z_2)^{1/2}\right] \omega & \omega \geq \omega^* \\ (z_2 K_2)^{1/2} \left[(K_1 z_1)^{1/2} + (K_2 z_2)^{1/2}\right] (1-\omega) & \omega < \omega^* \end{cases}$$

$$k(\omega) = \begin{cases} \frac{K_1^{1/2}\left[(K_1 z_1)^{1/2} + (K_2 z_2)^{1/2}\right]}{z_1^{1/2}} & \omega \geq \omega^* \\ \frac{K_2^{1/2}\left[(K_1 z_1)^{1/2} + (K_2 z_2)^{1/2}\right]}{z_2^{12}} & \omega < \omega^* \end{cases}$$

$$C_1 = \left[K_1 z_1 + (K_1 z_1 K_2 z_2)^{1/2}\right] e^{-1}$$

$$C_2 = \left[K_2 z_2 + (K_1 z_1 K_2 z_2)^{1/2}\right] e^{-1}$$

Part (e)

The CE is similar to that in part (a), except that now we have a savings decision. Specifically, an autarkic CE for country j is household allocations $[C_j(t), K_j(t)]_{t=0}^{\infty}$ and allocations for intermediate firms $[x_j(\omega,t), k_j(\omega,t)]_{\omega\in[0,1]}^{t\in[0,\infty)}$, along with prices $[p_j(\omega,t), r_j(t)]_{\omega\in[0,1]}^{t\in[0,\infty)}$, such that

1. Households maximize utility:

$$\max_{[C_j(t),K_j(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\sigma}}{1-\sigma} \ s.t.$$

$$\dot{K}_j(t) \leq r_j K_j(t) - C_j(t)$$

$$K_j(t), C_j(t) \geq 0$$

$$K_j(0) \text{ given}$$

2. Intermediate firms maximize profits for all t:

$$\max_{k_j(\omega,t)} \ p_j(\omega,t) A_j(\omega) k_j(\omega,t) - r_j(t) k_j(\omega,t)$$

3. The final firm maximizes profits for all t:

$$\max_{x_j(\omega,t)} \ e^{\int_0^1 \log x_j(\omega,t) d\omega} - \int_0^1 p_j(x_j(\omega,t),t) x_j(\omega,t) d\omega$$

4. Markets clear at every t:

$$C_j(t) + \dot{K}_j(t) + \delta K_j(t) = Y_j(t)$$
$$x_j(\omega, t) = A_j(\omega) k_j(\omega, t)$$
$$K_j(t) = \int_0^1 k_j(\omega, t) d\omega$$

Part (f)

Let's start by taking some FOC's of the planner's problem. From the household:

$$\frac{\dot{\lambda}_j(t)}{\lambda_j(t)} = \delta - Y_j'(t)$$
$$e^{-\rho t}[C_j(t)]^{-\sigma} = \lambda_j(t)$$

This gives us the Euler equation

$$\frac{\sigma}{C_j(t)} \dot{C}_j(t) = Y_j'(t) - \delta - \rho$$
$$\implies \sigma \frac{\dot{C}_j(t)}{C_j(t)} = Y_j'(t) - \delta - \rho$$

Note how the use of CRRA preferences simplifies the problem; the risk-aversion term is no longer a function of $C_j(t)$. From part (b), we know that the marginal productivity of capital is z_j/e. Therefore at the BGP we will have

$$\sigma g_c = z_j/e - \delta - \rho$$
$$\implies g_c = \frac{z_j/e - \delta - \rho}{\sigma}$$

Part (g)

A CE is as above, except that now the firm's problem is as in part (c), and the market-clearing

$$x_i^1(\omega, t) + x_i^2(\omega, t) = A_i(\omega) k_i(\omega, t)$$

holds for every period; and we will again have common prices $p(\omega, t)$ for intermediate goods.

Part (h)

The results from part (d) still hold in each period:

$$x(\omega, t) = \begin{cases} (z_1 K_1(t))^{1/2} \left[(K_1(t) z_1)^{1/2} + (K_2(t) z_2)^{1/2}\right] \omega & \omega \geq \omega^* \\ (z_2 K_2(t))^{1/2} \left[(K_1(t) z_1)^{1/2} + (K_2(t) z_2)^{1/2}\right] (1 - \omega) & \omega < \omega^* \end{cases}$$

where

$$\omega^* = \frac{(K_2(t)z_2)^{1/2}}{(K_1(t)z_1)^{1/2} + (K_2(t)z_2)^{1/2}}$$

Part (i)

From part (d), we have that

$$\frac{\delta Y}{\delta K_1} = \left(z_1 + \frac{(z_1 z_2 K_2)^{1/2}}{K_1^{1/2}}\right) e^{-1}$$
$$\frac{\delta Y}{\delta K_2} = \left(z_2 + \frac{(z_1 z_2 K_1)^{1/2}}{K_2^{1/2}}\right) e^{-1}$$

Hence the Euler equations for the two countries are

$$\frac{\dot{C}_1(t)}{C_1(t)} = \frac{1}{\sigma}\left(\frac{z_1}{e} + \frac{(z_1 z_2 K_2(t))^{1/2}}{K_1^{1/2}(t)e} - \delta - \rho\right)$$
$$\frac{\dot{C}_2(t)}{C_2(t)} = \frac{1}{\sigma}\left(\frac{z_2}{e} + \frac{(z_1 z_2 K_1(t))^{1/2}}{K_2^{1/2}(t)e} - \delta - \rho\right)$$

Note first of all that growth rates are strictly larger with trade, than without. This gain is coming from the complementary nature of the production functions: each nation specializes in its strengths, and because capital growth is concentrated in industries where each country has larger productivity, this in turn increases the marginal product of capital and allows for larger growth rates in consumption and investment.

Secondly, note that if K_1 and K_2 don't grow at the same rate, then consumption cannot grow at a constant rate. Hence, the BGP specifies not only a rate of growth, but also a relative size of K_1 to K_2, solving the system of equations

$$g = \frac{1}{\sigma}\left(\frac{z_1}{e} + \frac{(z_1 z_2)^{1/2}}{e} S^{-1/2} - \delta - \rho\right)$$
$$g = \frac{1}{\sigma}\left(\frac{z_2}{e} + \frac{(z_1 z_2)^{1/2}}{e} S^{1/2} - \delta - \rho\right)$$

where S is the size of K_1 relative to K_2.

Problem 5

Part (a)

A tax distorted competitive equilibrium in this economy is defined by time paths of consumption level, aggregate spending on intermediate goods, and aggregate R&D expediture $[C_t, X_t, Z_t]_{t=0}^{\infty}$, time paths of intermediate good varieties $[N_t]_{t=0}^{\infty}$, time paths of prices and quantities of each intermediate good $[p_t(i), x_t(i)]_{i\in[0,N_t],t=0}^{\infty}$, and time paths of prices $[r_t, w_t]_{t=0}^{\infty}$, such that given government policy time paths $[\tau_t^a, s_t, T_t]_{t=0}^{\infty}$ (where τ_t^k are taxes on savings income, s_t are subsidies on product creation and T_t are lump-sum taxes on households), households maximize utility, firms maximize discounted value of profits, the evolution of $[N_t]_{t=0}^{\infty}$ is determined by free entry the government budget constraint holds, and markets clear. That is, households solve

$$\max_{c_t,k_t} \int_0^{\infty} e^{\rho t} u(c_t) dt$$

subject to

$$\dot{a}_t = (r_t)(k_t)(1-\tau_t^a) + w_t - c_t - T_t$$

$$a_0 : given$$

Intermediate good producing firms solve

$$\max_{p_t(i),x_t(i)} p_t(i)x_t(i) - \psi x_t(i)$$

And $[N_t]_{t=0}^{\infty}$ evolves according to $\dot{N}_t = \eta Z_t$ and the free entry condition implies $\eta V(t) = \kappa - s_t\eta$ where

$$V_t = \int_t^{\infty} e^{-r(s-t)}\pi_s ds$$

Part (b)

We can formulate the current value Hamiltonian

$$H(c_t, a_t, \lambda_t) = u(c_t) + \lambda_t[r_t a_t(1-\tau_t^a) + w_t - c_t - T_t]$$

Necessary and sufficient conditions for optimum are

$$u'(c_t) = \lambda_t$$
$$\dot{\lambda}_t = \lambda_t(\rho - (r_t(1-\tau_t^a))$$
$$\dot{a}_t = (r_t)(k_t)(1-\tau_t^a) + w_t - c_t - T_t$$

Which yield the Euler equation

$$\frac{\dot{c}_t}{c_t} = \frac{u'(c_t)}{u''(c_t)c_t}(\rho - r_t(1-\tau_t^a)$$
$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}(\rho - r_t(1-\tau_t^a)$$

The free entry condition implies that

$$\eta V_t = \kappa - s_t \eta$$

Part (c)

On a balanced growth path $\frac{\dot{Y}_t}{Y_t} = \frac{\dot{C}_t}{C_t} = \frac{\dot{N}_t}{N_t} = \frac{\dot{Z}_t}{Z_t}$ Assuming taxes and subsidies are constant we now have

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}(r_t(1-\tau^a) - \rho)$$

and

$$\eta V_t = \kappa - s\eta$$

Now consider what the real interest rate must be in this economy. In class we derived the value of profits for the entrepreneur problem which remains the same here. Using this derivation and the fact that $\eta V_t = \kappa - s\eta$ We can see that

$$\begin{aligned}
\eta V_t &= \kappa - s\eta \\
\eta \frac{\psi(1-\alpha)x^*}{\alpha r^*} &= \kappa - s\eta \\
r^* &= \frac{\eta(\psi(1-\alpha)}{(\kappa - s\eta)\alpha} x^* \\
r^* &= \frac{\eta(\psi(1-\alpha)}{(\kappa - s\eta)\alpha} \left(\frac{\psi}{\alpha^2}\right)^{\frac{1}{\alpha-1}} L
\end{aligned}$$

Plugging into our euler equation above we see,

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left(\frac{\eta(\psi(1-\alpha)}{(\kappa - s\eta)\alpha} \left(\frac{\psi}{\alpha^2}\right)^{\frac{1}{\alpha-1}} L(1-\tau^a) - \rho\right)$$

Part (d)

Clearly from the above we see that an increase in τ^a would lower the growth rate on the BGP. It is also clear that an increase in the subsidy of successful inventions, s, would increase the growth rate along the BGP.

Part (e)

Assuming the CRRA utility function used in class, welfare maximizing level of taxes would maximize the growth rate $\dot{c}_t/c_t$ along the balanced growth path. Therefore we want to solve

$$\max_{s,\tau^a} \frac{1}{\sigma}(r_t(1-\tau^a) - \rho)$$

subject to

$$s\eta = \tau^a r_t^*$$

which is the government budget constraint. Plugging the constraint in,

$$\max_{s} \frac{1}{\sigma}(r_t - s\eta - \rho)$$

. Now substituting back in r_t

$$\max_{s} \frac{1}{\sigma}(\frac{\eta(\psi(1-\alpha)}{(\kappa - s\eta)\alpha}\left(\frac{\psi}{\alpha^2}\right)^{\frac{1}{\alpha-1}} L - s\eta - \rho)$$

Taking the first order condition,

$$\frac{1}{\sigma}\eta\alpha(\eta(\psi(1-\alpha)[(\kappa - s\eta)\alpha]^{-2}(\frac{\psi}{\alpha^2})^{\frac{1}{\alpha-1}} L - \frac{\eta}{\sigma} = 0$$

Let

$$B = \frac{1}{\sigma}\eta(\psi(1-\alpha)(\frac{\psi}{\alpha^2})^{\frac{1}{\alpha-1}} L$$
$$D = \frac{\eta}{\sigma}$$

We then have

$$\frac{B}{(\kappa\alpha - s\eta\alpha)^2} - D = 0$$
$$(\kappa\alpha - s\eta\alpha)^2 = \frac{B}{D}$$
$$s = \frac{\kappa\alpha - \sqrt{\frac{B}{D}}}{\eta\alpha}$$

Note that

$$\frac{B}{D} = \psi(1-\alpha)(\frac{\psi}{\alpha^2})^{\frac{1}{\alpha-1}} L$$
$$= (1-\alpha)\psi^{1+\frac{1}{(\alpha-1)}}\alpha^{\frac{-2}{\alpha-1}}$$
$$\implies \sqrt{\frac{B}{D}} = \alpha\sqrt{\left(\frac{1}{\alpha^2}(1-\alpha)\psi^{1+\frac{1}{(\alpha-1)}}\alpha^{\frac{-2}{\alpha-1}}\right)}$$

Therefore subsidies (and therefore government taxation) should be positive as long as

$$\kappa > \sqrt{\frac{1}{\alpha^2}(1-\alpha)\psi^{1+\frac{1}{(\alpha-1)}}\alpha^{\frac{-2}{\alpha-1}}}$$

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