Problem 1

Part (a)

We are given that the production function is Cobb-Douglas, and we are also given that labor supply is

$$\begin{aligned}\hat{N}_t &= N_t(\bar{e} - bn_t) \\ &= N_0 \prod_{i=0}^{t-1} n_i(\bar{e} - bn_t)\end{aligned}$$

Given this definition for $\hat{N}$, aggregate production will be

$$Y_t = K_t^{\alpha} \hat{N}_t^{1-\alpha}$$

and so we can write the feasibility constraint as

$$\begin{aligned}C_t + X_t &\leq K_t^{\alpha} \hat{N}_t^{1-\alpha} \\ K_{t+1} &= X_t + (1-\delta)K_t\end{aligned}$$

where $C_t, K_{t+1} \geq 0$ and $\delta \in [0,1]$.

Part (b)

I assume for simplicity that the consumption and investment goods are identical, and normalize $p_t = 1$. Let $N_t^i = N_0^i \prod_{i=0}^{t-1} n_i$ where N_t^i is the initial size of dynasty i. On the income side of the dynasty's budget constraint, we will have wages of $N_{t-1}^i(\bar{e} - bn_t^i)w_t$ and rent of $k_t^i r_t$, as well as bond holdings from the previous period a_{t-1}. On the expense side we will have nominal consumption expenditure of c_t^i and investment x_t^i and bond purchases of $a_t q_t$. Hence we can write the dynasty's budget constraint as

$$c_t^i + x_t^i + a_t q_t \leq w_t(\bar{e} - bn_t^i)N_t^i + r_t k_t^i + a_{t-1}$$

Part (c)

If all households are identical, then we can write $c_t^i = c_t = C_t/N_t$. A CE is allocations for dynasties $\{c_t, x_t, k_t, n_t\}_{t=0}^{\infty}$ and the representative firm $\{Y_t^f, K_t^f, L_t^f\}_{t=0}^{\infty}$ and (relative) prices $\{p_t = 1, r_t, w_t\}_{t=0}^{\infty}$ such that

1. Dynasties solve the optimization problem

$$\max_{\{c_t,x_t,k_{t+1},n_t\}_{t=0}^{\infty}} u(c_0)+\sum_{t=1}^{\infty}\beta^t u(c_t)\prod_{i=0}^{t-1}g(n_i)\ s.t.$$
$$\sum_{t=0}^{\infty}[c_t+x_t]\leq\sum_{t=0}^{\infty}[w_t(\bar{e}-bn_t)N_t+r_tk_t]$$
$$k_{t+1}=x_t+(1-\delta)k_t$$
$$N_t=N_{t-1}n_{t-1} \qquad c_t,n_t,k_{t+1}\geq 0$$
$$k_0 \text{ given}$$

2. The representative firm maximizes profits:

$$\max_{K_t^f,L_t^f} Y_t^f-r_tK_t^f-w_tL_t^f\ s.t.$$
$$Y_t^f=(K_t^f)^{\alpha}(L_t^f)^{1-\alpha}$$

3. Markets clear:

$$\text{Goods market}: \quad C_t+X_t=Y_t^f$$
$$\text{Capital market}: \quad K_t=K_t^f$$
$$\text{Labor market}: \quad (\bar{e}-bn_t)N_t=L_t^f$$

Part (d)

Making the usual assumptions about $u(\cdot)$, and putting everything in per capita terms, we can write the planner's problem as

$$\max_{c_t,k_{t+1},n_t} u(c_0)+\sum_{t=1}^{\infty}\beta^t u(c_t)\prod_{i=0}^{t-1}g(n_i)\ s.t.$$
$$c_t+n_tk_{t+1}=k_t^{\alpha}[(\bar{e}-bn_t)N_t]^{1-\alpha}+(1-\delta)k_t$$
$$N_{t+1}=N_tn_t$$
$$c_t,k_{t+1}\geq 0$$
$$k_0 \text{ given} \tag{1}$$

Now since the production function is homogeneous of degree one, we have

$$F(K_t,\hat{N}_t)=F_k(K_t,\hat{N}_t)K_t+F_n(K_t,\hat{N}_t)\hat{N}_t$$

and we know that at any competitive equilibrium, we will have

$$r_t k_t = F_k(K_t, \hat{N}_t) K_t$$
$$w_t(\bar{e} - bn_t) N_t = F_n(K_t, \hat{N}_t) \hat{N}_t$$

Substituting this into the feasibility constraint and writing investment in terms of X, we have

$$c_t + x_t = w_t(\bar{e} - bn_t) N_t + r_t k_t$$

which is the same as the dynasty's budget constraint. This implies, since all dynasties are the same, that the prices $\{p_t = 1, w_t, r_t\}_{t=0}^{\infty}$ ensure that any solution to the planner's problem is a competitive equilibrium.

Part (e)

Really this question is asking, what do we need to do to ensure that an interior solution exists? Taking FOC w.r.t. n_0 of (1)

$$\Big(- k_1 - bF_n(k_t, n_t) \Big) c_0^{-\sigma} + \sum_{t=1}^{\infty} \beta^t u(c_t) \left(\prod_{i=0}^{t-1} g(n_i) \frac{g'(n_0)}{g(n_0)} \right) = 0$$

The first term is always negative, as is $u(c_t)$. Hence, for there to be any hope of an interior solution, we need $g'(\cdot)$ to be less than zero as well (note that $g(\cdot) < 0$ won't work). As Hakki points out, setting $g'(0) = -\infty$ ensures a positive number of children, and the solution will be interior since the left-most term goes to negative infinity as $bn_0 \to \bar{e}$, so that at some (unique) point in $(0, \bar{e}/b)$ the equality will hold.

Part (f)

Note that in per capita terms, we need to adjusts next-period capital to reflect the increasing population. The recursive problem can be written as

$$V(k, N) = \max_{k' \in \mathbb{R}^+, n \in [0, \bar{e}/b]} \Big(u\big[k^\alpha (\bar{e} - bn)^{1-\alpha} + (1-\delta)k - nk' \big] + g(n) \beta V(k', Nn) \Big)$$

The lack of any role for the state variable N comes from the problem's assumption about the functional form of $g(\cdot)$ (i.e. that n enters as a divisor). Hence we can write this as

$$V(k) = \max_{k' \in \mathbb{R}^+, n \in [0, \bar{e}/b]} \Big(u\big[k^\alpha (\bar{e} - bn)^{1-\alpha} + (1-\delta)k - nk' \big] + g(n) \beta V(k') \Big)$$

Part (g)

Substituting these assumptions into the objective function, we have

$$\frac{c_0^{1-\sigma}}{1-\sigma}+\sum_{t=1}^{\infty}\beta^t\frac{c_t^{1-\sigma}}{1-\sigma}\prod_{i=0}^{t-1}n_i^{1+\eta}=\frac{c_0^{1-\sigma}}{1-\sigma}+\sum_{t=1}^{\infty}\beta^t N_t^{1+\eta}\frac{c_t^{1-\sigma}}{1-\sigma}$$

since $c_t = C_t/N_t$. Consider the FOC with respect to n_0:

$$\Big(-k_1-bF_n(k_t,n_t)\Big)c_0^{-\sigma}+\sum_{t=1}^{\infty}\beta^t N_t^{1+\eta}\frac{(1+\eta)}{n_0}\frac{c_t^{1-\sigma}}{1-\sigma}=0$$

For there to be an interior solution, we require that

$$\frac{1+\eta}{1-\sigma}>0$$

Therefore either $\sigma\in(0,1)$ and $\eta>-1$, or $\sigma>1$ and $\eta<-1$. If these conditions are met then we will certainly have an interior solution, since the RH term 'blows up' as $n_0\to 0$, and the LH term blows up as $n_0\to\bar{e}/b$. We can also obtain these results by considering the FOC w.r.t. fertility for the recursive problem, which recall has only capital as a state variable:

$$c^{-\sigma}(bF_k+k')=(1+\eta)n^{\eta}\beta V(k')$$

The LHS is positive; $V(k)$ is negative if $\sigma>1$, and positive if $\sigma<1$, which in turn gives us the necessary conditions on η for an interior solution.

Part (h)

Writing the planner's problem as

$$\max_{\{k_{t+1},n_t\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t N_t^{\sigma+\eta}\frac{\big(K_t^{\alpha}(\bar{e}-bn)^{1-\alpha}N_t^{1-\alpha}+(1-\delta)K_t-K_{t+1}\big)^{1-\sigma}}{1-\sigma}$$

$$c_t,k_{t+1}\geq 0$$

$$k_0 \text{ given}$$

we want to first take FOC's

$$\text{Capital}:\quad C_t^{-\sigma}=\beta n_t^{\sigma+\eta}C_{t+1}^{-\sigma}\Big(F_K(K_{t+1},\hat{N}_{t+1})+(1-\delta)\Big)$$

$$\text{Fertility}:\quad bF_N(K_t,\hat{N}_t)C_t^{-\sigma}=\beta\left[(\sigma+\eta)n_t^{\sigma+\eta-1}\frac{C_{t+1}^{1-\sigma}}{1-\sigma}+n_t^{\sigma+\eta}F_N(K_{t+1},\hat{N}_{t+1})C_{t+1}^{-\sigma}\right]$$

Now at any BGP we will have $c_{t+1} = c_t$ and likewise for per capita capital and labor supply, which implies that aggregate variables grow at the same rate as population. Therefore, considering the Euler equation for capital, at any BGP we will have

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = n^{\sigma}$$
$$\implies 1 = \beta n^{\eta}(F_K + 1 - \delta)$$

noting that F_k will be constant given that K and N are growing at equal rates. Now considering the Euler equation for fertility,

$$bF_n C^{-\sigma} = \beta\left[(\sigma+\eta)n^{\eta}\frac{C^{1-\sigma}}{1-\sigma} + n^{\eta}F_N C^{-\sigma}\right]$$
$$\implies bF_n = \beta n^{\eta}\left[\frac{\sigma+\eta}{1-\sigma}C + F_N\right]$$
$$\implies bF_N = \beta n^{\eta}\left[\frac{\sigma+\eta}{1-\sigma}c + F_N\right]$$

where in the last step I have switched to per capita terms. Together with the dynasty's budget constraint, these Euler equations characterize the BGP.

Part (i)

Note the typo: $-\sigma = \eta$ instead of $1 - \sigma = \eta$. With this substitution the Euler equation for fertility becomes

$$b = \beta n^{\eta}$$

which implies that

$$n = \left(\frac{b}{\beta}\right)^{\frac{1}{\eta}}$$

Rewriting $\hat{N}_t = (\bar{e} - bn)H_t$ and writing the law of motion for H_t as

$$H_{t+1} = nH_t$$

we can view population increase as investment in human capital (without depreciation). This is similar to the AKH model of growth.

Part (j)

Recall that at the BGP, per capita variables are constant so *all* growth in the economy is coming through population growth. In reality, we see growth in per capita variables as well. This implies that productivity is improving over time - a dynamic missing from this model. Hakki makes some good points on last year's solutions.

Problem 2

Part (a)

A Pareto optimal allocation is one for which we cannot make a Pareto improvement - that is, we cannot make any individual strictly better off without making another individual strictly worse off. More formally, if we assume that agents' utility functions are strictly concave, then any allocation satisfying the problem

$$\begin{aligned}\max_{\{c_t^i, k_{t+1}^i\}_{t=0}^{\infty}} \; & \sum_{t=0}^{\infty} \alpha_i u(c_t^i) \; s.t. \\ & \sum_i \left(c_t^i + k_{t+1}^i\right) = F(K_t, K_t LB) + \sum_i (1-\delta) k_t^i \\ & \sum_i \alpha_i = 1 \\ & c_t, k_{t+1}^i \geq 0 \; \forall \; i, t\end{aligned}$$

is Pareto optimal.

Part (b)

Letting U indicate aggregate utility (ignoring for simplicity how individual households are weighted), we can find the set of Pareto optimal allocations for this model by solving the planner's problem

$$\begin{aligned}\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} \; & \sum_{t=0}^{\infty} \beta^t U(C_t) \; s.t. \\ & C_t + K_{t+1} = K_t (BL)^{1-\alpha} + (1-\delta) K_t \\ & C_t, K_{t+1} \geq 0\end{aligned}$$

where B is the externality coefficient and L is aggregate labor. Taking FOC,

$$U'(C_t) = \beta U'(C_{t+1})(BL^{1-\alpha} + 1 - \delta)$$

which gives us the Euler equation for consumption.

Part (c)

A TDCE is allocations for the households $\{c_t^i, x_t^i, k_{t+1}^i\}_{t=0}^{\infty}$ and firms $\{k_t^j, n_t^j\}_{t=0}^{\infty}$, prices $\{r_t, w_t\}_{t=0}^{\infty}$ (noting that we can normalize the price of the consumption/investment good to 1), and policies $\{G_t = 0, T_t^i, \tau_{k,t}\}_{t=0}^{\infty}$ such that

1. Households solve the problem

$$\max_{c_t^i, k_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \ s.t.$$

$$c_t^i + x_t = (1 - \tau_{k,t}) r_t k_t^i + w_t \bar{l}_t + T_t^i$$

$$k_{t+1}^i = x_t^i + (1 - \delta) k_t^i$$

$$c_t^i, k_{t+1}^i \geq 0$$

Note that the capital tax affects *undepreciated* capital. Sometimes it will be more convenient to write capital taxes as net of depreciation (i.e. $(1 - \tau_{k,t})(r_t + 1 - \delta)k_t$), but here things will work out simpler if taxes are proportional to r_t. Note also that I've written the budget constraint sequentially - basically a trade-off between constant prices or a constant Lagrange multiplier.

2. Firms solve

$$\max_{k_t^j, n_t^j} (k_t^j)^{\alpha} (A_t n_t^j)^{1-\alpha} - r_t k_t^j - w_t n_t^j$$

3. Markets clear:

$$\sum_i c_t^i + x_t^j = \sum_j y_t^j \qquad \text{(defining } x_t^j \text{ and } y_t^j \text{ as usual)}$$

$$\sum_i k_t^i = \sum_j k_t^j$$

$$\sum_i \bar{l} = L = \sum_j n_t^j \qquad \forall t$$

4. Government budget balances:

$$\sum_t \sum_i T_t^i = \sum_t \sum_i \tau_{k,t} r_t k_t^i$$

Part (d)

For the TDCE, we have the FOC's:

$$\begin{aligned} c_t: \quad & \beta^t u'(c_t) = \lambda_t \\ k_{t+1}: \quad & \lambda_t = \left[(1-\tau_{k,t})r_t + 1 - \delta\right]\lambda_{t+1} \\ k_t^f: \quad & r_t = \alpha(k_t^j)^{\alpha-1}(A_t n_t^j)^{1-\alpha} = \alpha(K_t)^{\alpha-1}(BLK_t)^{1-\alpha} = \alpha BL^{1-\alpha} \end{aligned}$$

which we can combine to get the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})\left[(1-\tau_{k,t})\alpha BL^{1-\alpha} + 1 - \delta\right]$$

Compare this with the Euler equation from the planner's problem:

$$U'(C_t) = \beta U'(C_{t+1})(BL^{1-\alpha} + 1 - \delta)$$

It is evident that for these to coincide, we will need $(1-\tau_{k,t}) = \frac{1}{\alpha}$. But since $\alpha < 1$, it must be that $\tau_{k,t} < 0$, indicating that for the government to achieve a Pareto optimal allocation it must *subsidize* capital. This is because, from the standpoint of an individual household, the productivity term A_t is fixed: it will be affected only infinitesimally by the household's decisions, and so the household doesn't take into account the positive externality that happens when all households increases their investment. By subsidizing capital income, the government essentially forces households to take this externality into account.

Problem 3

This solution is a little different from the one in the solutions manual, which doesn't quite match up with the book.

Part (a)

An equilibrium in this economy is allocations for the household that consist of consumption levels, machine expenditures, and research expenses $[C(t), X(t), Z(t)]_{t=0}^{\infty}$; allocations for the research firms $[p^x(\nu,t), x(\nu,t)]_{t=0}^{\infty}$; prices $[r(t), w(t)]_{t=0}^{\infty}$; and time path of aggregate technology $[N(t)]_{t=0}^{\infty}$, such that 1) households maximize utility, 2) firms maximize profits, 3) markets clear, and 4) aggregate technology is consistent with free entry and determined by by the LoM

$$\dot{N}(t) = \eta N(t)^{-\phi} Z(t)$$

As in the original problem, monopolists set prices

$$p^x(\nu,t) = \frac{\gamma}{1-\beta} = 1$$

and sell quantities

$$x(\nu,t) = L$$

for profits

$$\pi(\nu,t) = \beta L$$

Also as in the original problem, wages are given by

$$\begin{aligned}
&D_L \frac{1}{1-\beta}\left(\int_0^{N(t)} x(\nu,t)^{1-\beta} d\nu\right) L^\beta - D_L \int_0^{N(t)} P^x(\nu,t)x(\nu,t)d\nu \\
&= D_L \frac{1}{1-\beta}\left(\int_0^{N(t)} L^{1-\beta} d\nu\right) L^\beta - D_L \int_0^{N(t)} L d\nu \\
&= \frac{1}{1-\beta} N(t) - N(t) \\
&= \frac{\beta}{1-\beta} N(t)
\end{aligned}$$

Finally, letting $V(\nu,t)$ indicate the value of an intermediate firm selling good ν at time t, free entry implies that

$$\eta N(t)^{-\phi} V(\nu,t) \leq 1 \tag{2}$$

i.e. intermediate firms enter until the return on innovation (measured in return on a unit of input) is less than or equal to 0.

Part (b)-(c)

Using the results above, we can write the HBJ equation for an intermediate firm as

$$\begin{aligned}
r(t)V(\nu,t) &= \pi(\nu,t) + \dot{V}(\nu,t) \\
&= \beta L + g_L V(\nu,t) \\
\implies V(\nu,t) &= \frac{\beta L}{r(t) - g_L}
\end{aligned}$$

where g_L is the growth rate of labor, and hence profits since $\pi = \beta L$, at the steady state. Assuming an interior solution where (2) holds with equality, we have

$$\begin{aligned} N(t) &= (\eta V(\nu, t))^{\frac{1}{\phi}} \\ &= \left(\frac{\eta \beta L}{r(t) - g_L} \right)^{\frac{1}{\phi}} \end{aligned}$$

But as in the book, we require a constant interest rate $r(t) = r^*$, so that

$$N^* = \left(\frac{\eta \beta L}{r^* - g_L} \right)^{\frac{1}{\phi}}$$

is constant if $g_L = 0$. Hence, without population growth, we will not have technological growth and we will therefore not have economic growth. If on the other hand $g_L = \dot{L} > 0$, then we will have

$$\begin{aligned} D_t N^* &= D_t L^{\frac{1}{\phi}} \left(\frac{\eta \beta}{r^* - \dot{L}} \right)^{\frac{1}{\phi}} \\ \implies \frac{\dot{N}^*}{N^*} &= \frac{1}{\phi} \frac{\dot{L}}{L} \\ &= \frac{n}{\phi} \end{aligned}$$

showing growth in technology (and hence consumption) along the balanced growth path.

Problem 4

Part (a)

Let the TFP for country $i \in \{1, 2\}$ be given by A^i. Normalizing the price of the final good to $p(t) = 1$, a competitive equilibrium is allocations for the representative households $[c^i(t), k^i(t)]_{t=0}^{\infty}$ and firms $[n_f^i(t), k_f^i(t)]_{t=0}^{\infty}$ and prices $[w^i(t), r^i(t)]_{t=0}^{\infty}$ such that

1. Households solve the problem

$$\begin{aligned} \max_{[c^i(t), k^i(t)]_{t=0}^{\infty}} \int_{t=0}^{\infty} e^{-\rho t} u(c^i(t)) dt \ s.t. \\ c^i(t) + \dot{k}^i(t) = w^i(t) + [r^i(t) - \delta] k^i(t) \\ c^i(t), k^i(t) \geq 0 \ \forall \ t, i \\ k(0) \text{ given} \end{aligned}$$

2. Firms maximize profits:

$$\max_{n_f^i(t),k_f^i(t)} A^i F(k_f^i(t), n_f^i(t)) - r^i(t)k_f^i(t) - w^i(t)n_f^i(t)$$

3. Markets clear for all t:

$$\sum_i k^i(t) = \sum_i k_f^i(t)$$
$$1 = n_f^i(t) \ \forall \ i$$
$$\sum_i \left[c^i(t) + \dot{k}^i(t) + \delta k^i(t)\right] = \sum_i \left[A^i F(k_f^i(t), n_f^i(t))\right]$$

Part (b)

Pareto optimality is characterized by the planner's problem. Define $\alpha \in (0,1)$, and suppose that the planner's welfare weights are equal. Then planner's problem can be written as

$$\max_{[c(t),k(t)]_{t=0}^{\infty}} \int_{t=0}^{\infty} e^{-\rho t} u(c(t))dt \ s.t.$$
$$c(t) + \dot{k}(t) = A^1 F(\alpha k(t), 1) + A^2 F([1-\alpha]k(t), 1) - \delta k(t)$$
$$c(t), k(t) \geq 0 \ \forall \ t$$
$$k^i(0) \text{ given}$$

Note that the planner is choosing how to allocate both consumption and capital between the two economies, as well as the savings decision. For the rest of this problem I will assume the Inada conditions: $F(\cdot)$ is increasing and concave in both inputs, $\lim_{k\to 0} F_k(k,1) = \infty$ and $\lim_{k\to\infty} F_k(k,1) = 0$.

Part (c)

Writing this problem recursively,

$$\rho V(k) = \max_{\alpha,c} \left\{u(c) + \left[A^1 F(\alpha k, 1) + A^2 F([1-\alpha]k, 1) - \delta k - c\right]V'(k)\right\}$$

Note that we only need one state variable: $k(t)$. At any Pareto optimal outcome, capital will be allocated between the two countries so that the marginal product of capital is equal, and this will determine α (which will constant at the steady state).

Part (d)

First note that since rental rates are equalized across countries, we must have

$$A^1 F_k(\alpha k, 1) = A^2 F_k([1-\alpha]k, 1)$$

showing that if $A^1 > A^2$, then $\alpha > 1/2$ - i.e. the more productive country gets more capital. The consumption Euler equation is then

$$\sigma(c(t))\frac{\dot{c}(t)}{c(t)} = A^1 \alpha F_k(\alpha k, 1) + A^2[1-\alpha]F_k([1-\alpha]k, 1) - \delta - \rho$$

Therefore at the steady state where $\dot{c}(t)/c(t) = 0$,

$$\begin{aligned} 0 &= A^1 \alpha F_k(\alpha k_{ss}, 1) + A^2[1-\alpha]F_k([1-\alpha]k_{ss}, 1) - \delta - \rho \\ &= A^1 F_k(\alpha k_{ss}, 1) - \delta - \rho \\ &= A^2 F_k([1-\alpha]k_{ss}, 1) - \delta - \rho \end{aligned}$$

Note that the marginal productivity of capital (and hence the rental rate) is equalized across countries. Turning to consumption, we therefore have

$$\begin{aligned} c(t) &= w(t) + (\delta + \rho)k(t) - \delta k(t) \\ &= w(t) + \rho k(t) \end{aligned}$$

Part (e)

If we shut down trade, then we will in essence have two steady states - one for each country - given by

$$0 = A^i F_k(k_{ss}^i, 1) - \delta - \rho$$

where now we have to keep track of each country's capital stock. In the steady state, it is evident that capital stocks will converge to α and $1-\alpha$ as above, and that the amount of capital in each country will be the same as with trade.

But these two scenarios are not equivalent! Without trade, a 'rich' country cannot lend capital to a 'poor' country; instead, the poorer nation is forced to save its way to the steady state – we have effectively ruled out 'foreign direct investment.' This is initially bad for wage earners in poor countries, since it lowers their marginal product. Steady state capital stocks are not affected, but steady state ownership of capital is. Without trade, each nation will ultimately consume only what it produces, whereas with trade it is possible that rich nations stay rich, by virtue of their ownership of foreign capital.

Part (f)

Following on the discussion from part (e), we have two cases to consider.

Without trade, initial capital stocks do not affect long-run allocations. The loss of initial capital does imply that country 1 will take longer to converge to the steady state, which will have an effect on the progression of inequality; but broadly, we can say that inequality is 'baked in' to national productivity, and that it will persist only to the extent that one country is more productive than another.

But now consider the case with trade, and for simplicity let's ignore wage income. If one country begins with a larger share of capital, then its income is larger in the same proportion. Considering the Euler equation

$$\sigma(c(t))\frac{\dot{c}(t)}{c(t)} = r(t) - \delta - \rho$$

we can see that the precise effect of a larger income on savings rates will depend on the utility function; but in general, the initial inequality in wealth will persist as the wealthier country saves more (in absolute terms and, if the MPC is decreasing in wealth, then in relative terms, too). If utility is CRRA, then growth/savings rates will be the same in both countries, and so inequality will persist absolutely - at the steady state, each country's share of rental income will be the same as it was in period 0. Hence, in this case country 1's share of income is lowered permanently by the loss of initial capital; whether this increases or decreases inequality depends on the initial distribution of capital.

Problem 5

Part (a)

I assume a representative agent for each country. In autarky, a competitive equilibrium for country j is consumption C_j and allocations for intermediate firms $[x_j(\omega), k_j(\omega)]_{\omega\in[0,1]}$, along with prices $[p_j(\omega), r_j]_{\omega\in[0,1]}$ (normalizing the price of the final good to one in each country), such that

1. Households maximize utility:

$$\max_{C_j} u(C_j) \; s.t.$$

$$C_j \leq r_j K_j$$

$$K_j \text{ given}$$

2. Intermediate firms maximize profits:

$$\max_{k_j(\omega)} \; p_j(\omega)A_j(\omega)k_j(\omega) - r_j k_j(\omega)$$

3. The final firm maximizes profits:

$$\max_{x_j(\omega)} \; e^{\int_0^1 \log x_j(\omega)d\omega} - \int_0^1 p_j(x_j(\omega))x_j(\omega)d\omega$$

4. Markets clear:

$$\log C_j = \log Y_j = \int_0^1 \log x_j(\omega)d\omega$$
$$x_j(\omega) = A_j(\omega)k_j(\omega)$$
$$K_j = \int_0^1 k_j(\omega)d\omega$$

Part (b)

Taking the FOC for an intermediate firm, we have

$$A_j(\omega)p_j(\omega) = r_j$$

Likewise, for the final firm we have

$$x_j(\omega)p_j(\omega) = Y_j$$

From the market-clearing condition for intermediate goods, it follows that

$$\begin{aligned} Y_j &= A_j(\omega)k_j(\omega)p_j(\omega) \\ &= r_j k_j(\omega) \\ &= r_j k_j \end{aligned}$$

indicating that an equal amount of capital is rented by all intermediate goods-producers, and that therefore given the assumption of a unit continuum of producers,

$$x_j(\omega) = A_j(\omega)K_j$$

The production function for the final good therefore becomes

$$Y_j = e^{\int_0^1 \log(A_j(\omega)K_j)d\omega}$$

Noting that the functional form assumptions for $A_j(\omega)$ imply that

$$\int_0^1 \log(A_1(\omega)K_1)d\omega = \int_0^1 \log(z_1\omega K_1)d\omega = \log(z_1K_1) + [x\log x - 1]_0^1 \approx \log(z_1K_1) - 1$$
$$\int_0^1 \log(A_2(\omega)K_2)d\omega = \int_0^1 \log(z_2[1-\omega]K_2)d\omega = \log(z_2K_2) + [x\log x - 1]_0^1 \approx \log(z_2K_2) - 1$$

we have

$$Y_1 = z_1K_1e^{-1}$$
$$Y_2 = z_2K_2e^{-1}$$

Finally, we can solve for prices and the rental rate of capital:

$$\begin{aligned} r_j &= Y_j/K_j \\ &= \begin{cases} z_1e^{-1} & i=1 \\ z_2e^{-1} & i=2 \end{cases} \\ p_j(\omega) &= Y_j/K_jA_j(\omega) \\ &\approx \begin{cases} \frac{1}{e\omega} & i=1 \\ \frac{1}{e(1-\omega)} & i=2 \end{cases} \end{aligned}$$

Note that the endpoints $\{0,1\}$ are problematic these last few steps; we really need to add the assumption that in country one, there is no producer for $\omega = 0$, whereas in country 2 there is no producer for $\omega = 1$.

Part (c)

With trade in intermediate goods, it is evident that both final goods-producing firms will use the same proportions of inputs. Hence a CE is the same as above, except that now the problem for final goods firm i is

$$\max_{x_1^i(\omega), x_2^i(\omega)} \quad e^{\int_0^1 [\log x_1^i(\omega) + \log x_2^i(\omega)]d\omega} - \int_0^1 [p_1(x_1^i(\omega))x_1^i(\omega) + p_2(x_2^i(\omega))x_2^i(\omega)]d\omega$$

and market-clearing for intermediate goods is now

$$x_i^1(\omega) + x_i^2(\omega) = A_i(\omega)k_i(\omega)$$

Additionally, the prices needed for market clearing are now $[p(\omega), p_1, p_2]_{\omega=0}^{\infty}$. Trade implies that prices for intermediate goods will equalize, and we also need to take into account that prices of final goods may be different between the two countries (although in fact, they won't be).

Part (d)

By assumption, $A_1(0) = 0$ and $A_1'(\omega) = z_1$, whereas $A_2(1) = 0$ and $A_2'(\omega) = 1 - \omega$. Hence, considering the term

$$\frac{A_2(\omega)}{A_1(\omega)} = \frac{z_1\omega}{z_2(1-\omega)}$$

we know that this term is 0 when $\omega = 0$, is strictly increasing in ω, and goes to $+\infty$ at the limit. Now since the FOC for intermediate-good producers is $p_j(\omega) = r_j/A_j(\omega)$, we have

$$\begin{aligned}\frac{p_1(\omega)}{p_2(\omega)} &= \frac{r_1}{r_2} \times \frac{A_2(\omega)}{A_1(\omega)} \\ &= \frac{r_1}{r_2} \times \frac{z_1\omega}{z_2(1-\omega)}\end{aligned}$$

From the previous result, given any $r_1, r_2 \in \mathbb{R}^2_{++}$, there will exist a unique ω^* such that

$$r_1 A_2(\omega^*) = r_2 A_1(\omega^*)$$

and we will have $p_1(\omega') < p_2(\omega')$ for $\omega' > \omega^*$, and $p_1(\omega'') > p_2(\omega'')$ for $\omega'' < \omega^*$. Since $p_1(\omega)$ and $p_2(\omega)$ are perfect substitutes for any ω, it is evident that the firm will only purchase the lower-priced alternative, which in turn implies that it will not be profitable for intermediate firms in country 1 to produce for $\omega < \omega^*$, whereas intermediate firms in country 2 will not produce for $\omega > \omega^*$.

The FOC for firms are basically unchanged from above:

$$A_j(\omega)p(\omega) = r_j$$
$$x_i(\omega)p(\omega)p_i = Y_i$$

and substituting these into the market-clearing condition with trade,

$$Y_i = \begin{cases} r_1(\omega)k_1(\omega)p_i & \omega \geq \omega^* \\ r_2(\omega)k_2(\omega)p_i & \omega < \omega^* \end{cases}$$

Hence, in this case demand for intermediate goods is given by

$$x_1(\omega) + x_2(\omega) = \begin{cases} \frac{A_1(\omega)K_1}{1-\omega^*} & \omega \geq \omega^* \\ \frac{A_2(\omega)K_2}{\omega^*} & \omega < \omega^* \end{cases}$$

Now note that the production function for final goods is homogeneous of degree 1, so that the only difference between the two firms is that the producer for country 1 will be allocated some portion

α of both inputs and production, and the other firm will receive and produce a portion $1-\alpha$. But then the firms face the same marginal cost, implying that in this economy, equalization of intermediate goods prices implies equalization of final goods prices, which we can normalize to 1. Final production for country i is therefore given by

$$\begin{aligned}\log(Y_1+Y_2) &= \int_{\omega^*}^{1}\log\left[A_1(\omega)\frac{K_1}{1-\omega^*}\right]d\omega + \int_0^{\omega^*}\log\left[A_2(\omega)\frac{K_2}{\omega^*}\right]d\omega \\ &= \left[x\log\left(\frac{z_1K_1}{1-\omega^*}x\right)-x\right]_{\omega^*}^{1} + \left[x\log\left(\frac{z_2K_2}{\omega^*}x\right)-x\right]_{1-\omega^*}^{1} \\ &= \log\left(\frac{z_1K_1z_2K_2}{\omega^*(1-\omega^*)}\right) - \omega^*\log\left(\frac{\omega^*}{1-\omega^*}z_1K_1\right) - (1-\omega^*)\log\left(\frac{1-\omega^*}{\omega^*}z_2K_2\right) - 1\end{aligned}$$

At this point we can solve for prices,

$$\begin{aligned}p(\omega) &= \frac{Y}{x(\omega)} \\ &= \begin{cases}\frac{(1-\omega^*)z_1K_1z_2K_2(\omega^*[1-\omega^*])^{-1}\left(\frac{\omega^*}{1-\omega^*}z_1K_1\right)^{-\omega^*}\left(\frac{1-\omega^*}{\omega^*}z_2K_2\right)^{\omega^*-1}e^{-1}}{\omega z_1K_1} & \omega\geq\omega^* \\ \frac{w^*z_1K_1z_2K_2(\omega^*[1-\omega^*])^{-1}\left(\frac{\omega^*}{1-\omega^*}z_1K_1\right)^{-\omega^*}\left(\frac{1-\omega^*}{\omega^*}z_2K_2\right)^{\omega^*-1}e^{-1}}{(1-\omega)z_2K_2} & \omega<\omega^*\end{cases} \\ &= \begin{cases}(\omega^*)^{-1}(z_1K_1)^{-\omega^*}(z_2K_2)^{\omega^*}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}\frac{1}{\omega} & \omega\geq\omega^* \\ (1-\omega^*)^{-1}(z_1K_1)^{1-\omega^*}(z_2K_2)^{\omega^*-1}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}\frac{1}{1-\omega} & \omega<\omega^*\end{cases}\end{aligned}$$

Hence

$$\begin{aligned}r_1 &= z_1(\omega^*)^{-1}(z_1K_1)^{-\omega^*}(z_2K_2)^{\omega^*}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1} \\ r_2 &= z_2(1-\omega^*)^{-1}(z_1K_1)^{1-\omega^*}(z_2K_2)^{\omega^*-1}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}\end{aligned}$$

This tells us, first of all, the value of ω^*, since from the beginning of this section

$$\begin{aligned}r_1A_2(\omega^*) &= r_2A_1(\omega^*) \\ \implies (1-\omega^*)^2z_2K_2 &= (\omega^*)^2z_1K_1 \\ \implies \omega^* &= \frac{(K_2z_2)^{1/2}}{(K_1z_1)^{1/2}+(K_2z_2)^{1/2}}\end{aligned}$$

But then we have also solved for relative rents, since

$$\begin{aligned}\frac{r_1}{r_2} &= \frac{z_1(\omega^*)^{-1}(z_1K_1)^{-\omega^*}(z_2K_2)^{\omega^*}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}}{z_2(1-\omega^*)^{-1}(z_1K_1)^{1-\omega^*}(z_2K_2)^{\omega^*-1}e^{-1}\left(\frac{1-\omega^*}{\omega^*}\right)^{2\omega^*-1}} \\ &= \frac{K_2(1-\omega^*)}{K_1\omega^*} \\ &= \frac{(K_2z_1)^{1/2}}{(K_1z_2)^{1/2}}\end{aligned}$$

Using these results we can solve for production:

$$\begin{aligned}
Y_1+Y_2 &= z_1K_1z_2K_2(\omega^*[1-\omega^*])^{-1}\left(\frac{\omega^*}{1-\omega^*}z_1K_1\right)^{-\omega^*}\left(\frac{1-\omega^*}{\omega^*}z_2K_2\right)^{\omega^*-1}e^{-1}\\
&= z_1K_1z_2K_2\frac{\left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right]^2}{(z_1K_1z_2K_2)^{1/2}}\left(\left(\frac{K_2z_2}{K_1z_1}\right)^{1/2}z_1K_1\right)^{-\omega^*}\left(\left(\frac{K_1z_1}{K_2z_2}\right)^{1/2}z_2K_2\right)^{\omega^*-1}e^{-1}\\
&= \left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right]^2e^{-1}
\end{aligned}$$

We can now solve for quantities and prices:

$$\begin{aligned}
p(\omega) &= \begin{cases}\frac{\left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right]}{(K_1z_1)^{1/2}}e^{-1}\frac{1}{\omega} & \omega\geq\omega^*\\ \frac{\left[(K_2z_2)^{1/2}+(K_2z_2)^{1/2}\right]}{(K_2z_2)^{1/2}}e^{-1}\frac{1}{1-\omega} & \omega<\omega^*\end{cases}\\
r_1 &= z_1^{1/2}\frac{\left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right]}{K_1^{1/2}}e^{-1}\\
r_2 &= z_2^{1/2}\frac{\left[(K_2z_2)^{1/2}+(K_2z_2)^{1/2}\right]}{K_2^{1/2}}e^{-1}\\
x(\omega) &= \begin{cases}(z_1K_1)^{1/2}\left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right]\omega & \omega\geq\omega^*\\ (z_2K_2)^{1/2}\left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right](1-\omega) & \omega<\omega^*\end{cases}\\
k(\omega) &= \begin{cases}\frac{K_1^{1/2}\left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right]}{z_1^{1/2}} & \omega\geq\omega^*\\ \frac{K_2^{1/2}\left[(K_1z_1)^{1/2}+(K_2z_2)^{1/2}\right]}{z_2^{12}} & \omega<\omega^*\end{cases}\\
C_1 &= K_1z_1+(K_1z_1K_2z_2)^{1/2}\\
C_2 &= K_2z_2+(K_1z_1K_2z_2)^{1/2}
\end{aligned}$$

Part (e)

The CE is similar to that in part (a), except that now we have a savings decision. Specifically, an autarkic CE for country j is household allocations $[C_j(t),K_j(t)]_{t=0}^{\infty}$ and allocations for intermediate firms $[x_j(\omega,t),k_j(\omega,t)]_{\omega\in[0,1]}^{t\in[0,\infty)}$, along with prices $[p_j(\omega,t),r_j(t)]_{\omega\in[0,1]}^{t\in[0,\infty)}$, such that

1. Households maximize utility:

$$\max_{[C_j(t),K_j(t)]_{t=0}^{\infty}}\int_0^{\infty}e^{-\rho t}\frac{C_t^{1-\sigma}}{1-\sigma}\ s.t.$$

$$\dot{K}_j(t)\leq r_jK_j(t)-C_j(t)$$

$$K_j(t),C_j(t)\geq 0$$

$$K_j(0)\text{ given}$$

2. Intermediate firms maximize profits for all t:

$$\max_{k_j(\omega,t)} \; p_j(\omega,t)A_j(\omega)k_j(\omega,t) - r_j(t)k_j(\omega,t)$$

3. The final firm maximizes profits for all t:

$$\max_{x_j(\omega,t)} \; e^{\int_0^1 \log x_j(\omega,t)d\omega} - \int_0^1 p_j(x_j(\omega,t),t)x_j(\omega,t)d\omega$$

4. Markets clear at every t:

$$C_j(t) + \dot{K}_j(t) + \delta K_j(t) = Y_j(t)$$
$$x_j(\omega,t) = A_j(\omega)k_j(\omega,t)$$
$$K_j(t) = \int_0^1 k_j(\omega,t)d\omega$$

Part (f)

Let's start by taking some FOC's of the planner's problem. From the household:

$$\frac{\dot{\lambda}_j(t)}{\lambda_j(t)} = \delta - Y_j'(t)$$
$$e^{-\rho t}[C_j(t)]^{-\sigma} = \lambda_j(t)$$

This gives us the Euler equation

$$\frac{\sigma}{C_j(t)}\dot{C}_j(t) = Y_j'(t) - \delta - \rho$$
$$\implies \sigma\frac{\dot{C}_j(t)}{C_j(t)} = Y_j'(t) - \delta - \rho$$

Note how the use of CRRA preferences simplifies the problem; the risk-aversion term is no longer a function of $C_j(t)$. From part (b), we know that the marginal productivity of capital is z_j/e. Therefore at the BGP we will have

$$\sigma g_c = z_j/e - \delta - \rho$$
$$\implies g_c = \frac{z_j/e - \delta - \rho}{\sigma}$$

Part (g)

A CE is as above, except that now the firm's problem is as in part (c), and the market-clearing

$$x_i^1(\omega,t) + x_i^2(\omega,t) = A_i(\omega)k_i(\omega,t)$$

holds for every period; and we will again have common prices $p(\omega,t)$ for intermediate goods.

Part (h)

The results from part (d) still hold in each period:

$$x(\omega,t)=\begin{cases}(z_1K_1(t))^{1/2}\left[(K_1(t)z_1)^{1/2}+(K_2(t)z_2)^{1/2}\right]\omega & \omega\geq\omega^*\\ (z_2K_2(t))^{1/2}\left[(K_1(t)z_1)^{1/2}+(K_2(t)z_2)^{1/2}\right](1-\omega) & \omega<\omega^*\end{cases}$$

where

$$\omega^*=\frac{(K_2(t)z_2)^{1/2}}{(K_1(t)z_1)^{1/2}+(K_2(t)z_2)^{1/2}}$$

Part (i)

From part (d), we have that

$$\frac{\delta Y}{\delta K_1}=\left(z_1+\frac{(z_1z_2K_2)^{1/2}}{2K_1^{1/2}}\right)e^{-1}$$
$$\frac{\delta Y}{\delta K_2}=\left(z_2+\frac{(z_1z_2K_1)^{1/2}}{2K_2^{1/2}}\right)e^{-1}$$

Hence the Euler equations for the two countries are

$$\frac{\dot{C}_1(t)}{C_1(t)}=\frac{1}{\sigma}\left(\frac{z_1}{e}+\frac{(z_1z_2K_2(t))^{1/2}}{2K_1^{1/2}(t)e}-\delta-\rho\right)$$
$$\frac{\dot{C}_2(t)}{C_2(t)}=\frac{1}{\sigma}\left(\frac{z_2}{e}+\frac{(z_1z_2K_1(t))^{1/2}}{2K_2^{1/2}(t)e}-\delta-\rho\right)$$

Note first of all that growth rates are strictly larger with trade, than without. This gain is coming from the complementary nature of the production functions: each nation specializes in its strengths, and because capital growth is concentrated in industries where each country has larger productivity, this in turn increases the marginal product of capital and allows for larger growth rates in consumption and investment.

Secondly, note that if K_1 and K_2 don't grow at the same rate, then consumption cannot grow at a constant rate. Hence, the BGP specifies not only a rate of growth, but also a relative size of K_1 to K_2, solving the system of equations

$$g=\frac{1}{\sigma}\left(\frac{z_1}{e}+\frac{(z_1z_2)^{1/2}}{2e}S^{-1/2}-\delta-\rho\right)$$
$$g=\frac{1}{\sigma}\left(\frac{z_2}{e}+\frac{(z_1z_2)^{1/2}}{2e}S^{1/2}-\delta-\rho\right)$$

where S is the size of K_1 relative to K_2.