Exercise 10.2. *The term structure and regime switching, donated by Rodolfo Manuelli*

Consider a pure exchange economy where the stochastic process for consumption is given by,

$$c_{t+1} = c_t \exp[\alpha_0 - \alpha_1 s_t + \varepsilon_{t+1}],$$

where

(i) $\alpha_0 > 0$, $\alpha_1 > 0$, and $\alpha_0 - \alpha_1 > 0$.

(ii) ε_t is a sequence of i.i.d. random variables distributed $N(\mu, \tau^2)$. Note: Given this specification, it follows that $E[e^{\varepsilon}] = \exp[\mu + \tau^2/2]$.

(iii) s_t is a Markov process independent from ε_t that can take only two values, $\{0, 1\}$. The transition probability matrix is completely summarized by

$$\begin{aligned} \text{Prob}[s_{t+1} &= 1 | s_t = 1] = \pi(1), \\ \text{Prob}[s_{t+1} &= 0 | s_t = 0] = \pi(0). \end{aligned}$$

(iv) The information set at time t,Ω_t, contains $\{c_{t-j}, s_{t-j}, \varepsilon_{t-j}; j \geq 0\}$.

There is a large number of individuals with the following utility function

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u(c) = c^{(1-\sigma)}/(1-\sigma)$. Assume that $\sigma > 0$ and $0 < \beta < 1$. As usual, $\sigma = 1$ corresponds to the log utility function.

a. Compute the "short-term" (one-period) interest rate.

b. Compute the "long-term" (two-period) interest rate measured in the same time units as the rate you computed in a. (That is, take the appropriate square root.)

c. Note that the log of the rate of growth of consumption is given by

$$\log(c_{t+1}) - \log(c_t) = \alpha_0 - \alpha_1 s_t + \varepsilon_{t+1}.$$

Thus, the conditional expectation of this growth rate is just $\alpha_0 - \alpha_1 s_t + \mu$. Note that when $s_t = 0$, growth is high and, when $s_t = 1$, growth is low. Thus, loosely speaking, we can identify $s_t = 0$ with the peak of the cycle (or good times) and $s_t = 1$ with the trough of the cycle (or bad times). Assume $\mu > 0$. Go as far as you can describing the implications of this model for the cyclical behavior of the term structure of interest rates.

d. Are short term rates pro- or countercyclical?

e. Are long rates pro- or countercyclical? If you cannot give a definite answer to this question, find conditions under which they are either pro- or countercyclical, and interpret your conditions in terms of the "permanence" (you get to define this) of the cycle.

Solution

a. We use the formula derived in chapter 10. Specifically:

$$\frac{1}{R_{1t}} = E_t\left(\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right) \tag{92}$$

$$\frac{1}{R_{1t}} = \beta \exp(-\sigma\alpha_0 + \sigma\alpha_1 s_t) E_t\left(\exp(-\sigma\varepsilon_{t+1})\right) \tag{93}$$

$$\frac{1}{R_{1t}} = \beta \exp(-\sigma\alpha_0 + \sigma\alpha_1 s_t - \sigma\mu + \sigma^2\tau^2/2), \tag{94}$$

where the last equality follows from the fact that $-\sigma\varepsilon_{t+1}$ is normal with mean $-\sigma\mu$ and variance $\sigma^2\tau^2$.

b. Using again the formula of chapter 10 we have :

$$(95)\frac{1}{R_{2t}^2} = E_t\left(\beta^2\left(\frac{c_{t+2}}{c_{t+1}}\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right)$$
$$(96)\frac{1}{R_{2t}^2} = \beta^2 E_t\left(\exp(-\sigma\alpha_0+\sigma\alpha_1 s_{t+1}+\sigma\varepsilon_{t+2})\right)\exp(-\sigma\alpha_0+\sigma\alpha_1 s_t+\sigma\varepsilon_{t+1})$$
$$(97)\frac{1}{R_{2t}^2} = \beta^2\exp\left(2(-\sigma\alpha_0-\sigma\mu+\sigma^2\tau^2/2)\right)E_t\left(\exp(\sigma\alpha_1(s_t+s_{t+1}))\right).$$

Observe that either $s_{t+1}=s_t$, or $s_{t+1}=1-s_t$. Therefore, we can write:

$$E_t\left(\exp(\sigma\alpha_1 s_{t+1})\right)=\exp(\sigma\alpha_1 s_t)\times\left[\pi(s_t|s_t)+\pi(1-s_t|s_t)\exp(\sigma\alpha_1(1-2s_t))\right].$$

This yields to the following two expressions for the long rate :

$$(98)\qquad \frac{1}{R_{2t}} = \frac{1}{R_{1t}}\left[\pi(s_t|s_t)+\pi(1-s_t|s_t)\exp(\sigma\alpha_1(1-2s_t))\right]^{1/2}$$
$$(99)\qquad \frac{1}{R_{2t}} = \beta\exp(-\sigma\alpha_0-\sigma\mu+\sigma^2\tau^2/2)$$
$$(100)\qquad \times\left[\pi(s_t|s_t)\exp(2\sigma\alpha_1 s_t)+\pi(1-s_t|s_t)\exp(\sigma\alpha_1)\right]^{1/2}.$$

c.,d. and e. Equation (98) implies that, at the peak $s_t=0$, the long rate is smaller than the sort rate : the term structure of interest rates is downwards slopping. The intuition goes as follows. In two periods, there is a positive probability of low growth. Therefore, "long term consumption" is relatively scarcer than "short term consumption". Its price should be higher. In other words, the long term interest rate is lower than the short term interest rate.
Conversely, at the trough $s_t=1$, the long term interest rate is higher than the short term interest rate : the term structure of interest rates is upwards slopping.

Short term interest rates are low when $s_t=1$ (trough) and high when $s_t=0$ (peak). Again, this is because when $s_t=1$, the growth rate of consumption is low. Tomorrow's good is relatively scarcer than if $s_t=1$. Therefore, tomorrow's good should have a higher price when $s_t=1$ than when $s_t=0$. In other words, the short term interest rate is low at a trough and high at a peak. In this precise sense, the short term interest rate is procyclical

Examination of equation (100) shows that long term interest rate is procyclical. Also, procyclicality is stronger if $\pi(s_t|s_t)$ is closer to one, i.e. if shocks are persistent.

Exercise 10.3. *Growth slowdowns and stock market crashes, donated by Rodolfo Manuelli*

Consider a simple one-tree pure exchange economy. The only source of consumption is the fruit that grows on the tree. This fruit is called dividends by the tribe inhabiting this island. The stochastic process for dividend d_t is described as follows: If d_t is not equal to d_{t-1}, then $d_{t+1} = \gamma d_t$ with probability π, and $d_{t+1} = d_t$ with probability $(1-\pi)$. If in any pair of periods j and $j+1$, $d_j = d_{j+1}$, then for all $t > j$, $d_t = d_j$. In words, the process – if not stopped – grows at a rate γ in every period. However, once it stops growing for one period, it remains constant forever on. Let d_0 equal one. Preferences over stochastic processes for consumption are given by

$$U = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u(c) = c^{(1-\sigma)}/(1-\sigma)$. Assume that $\sigma > 0$, $0 < \beta < 1$, $\gamma > 1$, and $\beta\gamma^{(1-\sigma)} < 1$.

a. Define a competitive equilibrium in which shares to this tree are traded.

b. Display the equilibrium process for the price of shares in this tree p_t as a function of the history of dividends. Is the price process a Markov process in the sense that it depends just on the last period's dividends?

c. Let T be the first time in which $d_{T-1} = d_T = \gamma^{(T-1)}$. Is $p_{T-1} > p_T$? Show conditions under which this is true. What is the economic intuition for this result? What does it say about stock market declines or crashes?

d. If this model is correct, what does it say about the behavior of the aggregate value of the stock market in economies that switched from high to low growth (e.g., Japan)?

Solution

a. First define the household's problem :

$$\max_{\{c_t(d^t)\}} \sum_{t\geq 0} \sum_{d^t} \beta^t u\left(c_t(d^t)\right), \tag{101}$$

subject to $c_t(d^t) + s_t(d^t)p_t(d^t) = s_t(d^t)d_t + s_{t-1}(d^{t-1})p_t(d^t)$ and with $s_{-1} = 1$. Observe that we assume that the tree price is "cum-dividend".

Definition 16. *An equilibrium is an allocation $\{c_t(d^t), s_t(d^t)\}_{t\geq 0}$ and a price process $\{p_t(d^t)\}_{t\geq 0}$ such that:*

(i) *Optimality: given price, the allocation solves the household's problem*

(ii) *Feasibility: markets clear, i.e. $c_t(d^t) \leq d_t$ for all d^t.*

Let's derive the first order conditions of the household problem. Attach multiplier $\mu_t(d^t)$ to node d^t budget constraint. The first order conditions are :

$$\begin{array}{lll} c_t(d^t): & \beta^t \pi(d^t) u'(c_t(d^t)) & = \mu_t(d^t) \\ s_t(d^t): & \mu_t(d^t) p_t(d^t) & = d_t + \sum_{d_{t+1}} \mu_{t+1}(d^t, d_{t+1}) p_{t+1}(d^t, d_{t+1}). \end{array}$$

Substituting the first equation into the second one gives the familiar Euler equation:

$$p_t = d_t + \beta E_t \left(\frac{u'(c_{t+1})}{u'(c_t)} p_{t+1} \right). \tag{102}$$

Imposing market clearing $c_t = d_t$ gives the pricing formula :

$$p_t = d_t + \beta E_t \left(\frac{u'(d_{t+1})}{u'(d_t)} p_{t+1} \right). \tag{103}$$

b. We guess and verify that the price of the tree is of the form $p_i d_t$, where $i = g$ if the growth process is not stopped and $i = s$ if the growth process is stopped.

If the growth process is stopped, then $c_{t+k} = d_t$ for all $k \geq 0$. Therefore $\beta^k \frac{u'(c_{t+k})}{u'(c_t)} = \beta^k$ and the (cum dividend) price of the tree is $\frac{d_t}{1-\beta}$. Thus :

$$p_s = 1/(1-\beta).$$

If the growth process is not stopped then two things can happen tomorrow. First, with probability π the economy grows. In this event $c_{t+1}/c_t = \gamma$ and the dividend of the tree is $d_{t+1} = \gamma d_t$. Second, with probability $1-\pi$ the economy stops growing. In this event $c_{t+1}/c_t = 1$ and the dividend of the tree is $d_{t+1} = d_t$. Thus, the (cum dividend) price of the tree at time t is

$$p_g d_t = d_t + \beta \left(\pi \gamma^{-\sigma} p_g \gamma d_t + (1-\pi) d_t/(1-\beta) \right).$$

Solving for p_g gives :

$$p_g = \left(1 - \beta\pi\gamma^{1-\sigma}\right)^{-1} \left[1 + \frac{\beta(1-\pi)}{1-\beta}\right]. \tag{104}$$

The price of the tree is Markov provided we expand the state to (d_t, d_{t-1}). Specifically, if $d_t = d_{t-1}$, then the price is $p_s d_t$. If $d_t \neq d_{t-1}$, then $d_t = p_g d_t$.

c. and d. In term of our notations, we need to find conditions under which $p_g > p_s$. Using the above expressions shows that the inequality is equivalent to $\gamma > 1$. The value of the aggregate stock market is the value of a claim to the economy output. In the event of a growth slowdown, the economy is expected to grow at a lower rate and, thus, the value of the stock market declines from $p_g d_t$ to $p_s d_t$. This can be interpreted as a stock market crash.

Exercise 10.4. *The term structure and consumption, donated by Rodolfo Manuelli*

Consider an economy populated by a large number of identical households. The (common) utility function is

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $0 < \beta < 1$, and $u(x) = x^{1-\theta}/(1-\theta)$, for some $\theta > 0$. (If $\theta = 1$, the utility is logarithmic.) Each household owns one tree. Thus, the number of households and trees coincide. The amount of consumption that grows in a tree satisfies

$$c_{t+1} = c^* c_t^{\varphi} \varepsilon_{t+1},$$

where $0 < \varphi < 1$, and ε_t is a sequence of i.i.d. log normal random variables with mean one, and variance σ^2. Assume that, in addition to shares in trees, in this economy bonds of all maturities are traded.

a. Define a competitive equilibrium.

b. Go as far as you can calculating the term structure of interest rates, $\tilde{R}_{jt}$, for $j = 1, 2, \ldots$.

c. Economist A argues that economic theory predicts that the variance of the log of short-term interest rates (say one-period) is always lower than the variance of long-term interest rates, because short rates are "riskier." Do you agree? Justify your answer.

d. Economist B claims that short-term interest rates, i.e., $j = 1$, are "more responsive" to the state of the economy, i.e., c_t, than are long-term interest rates, i.e., j large. Do you agree? Justify your answer.

e. Economist C claims that the Fed should lower interest rates because whenever interest rates are low, consumption is high. Do you agree? Justify your answer.

f. Economist D claims that in economies in which output (consumption in our case) is very persistent ($\varphi \approx 1$), changes in output (consumption) do not affect interest rates. Do you agree? Justify your answer and, if possible, provide economic intuition for your argument.

Solution

a. We first describe the household's problem. To simplify it, we assume that bonds of maturities $k = j, \ldots J$ are traded, for a fix $J \geq 1$. In any period, the agent chooses bond holding of various maturities B_{jt}. The price at time t of a zero coupon bond maturing j periods from now is written q_{jt} [1]. The corresponding gross interest rate is $R_{jt} \equiv (1/q_{jt})^{1/j}$. The representative agent maximize:

[1] This bond pays 1 for sure at time $t + j$.

$$E_0 \quad \sum_{t=0}^{+\infty} \beta^t u(c_t) \quad ,$$

subject to:

$$c_t + p_t s_t + \sum_{j=1}^{J} q_{jt} B_{jt} = s_t d_t + p_t s_{t-1} + B_{1t-1} + \sum_{j=1}^{J-1} q_{jt} B_{j+1,t-1},$$

and $B_{j,-1} = 0$ for all j. We have dropped the dependence on ε^t to simplify notations.

DEFINITION 17. *An equilibrium is an allocation* $\{c_t, s_t, B_{jt}\}_{t=0}^{+\infty}$*, a price process* $\{p_t, q_{jt}\}_{t=0}^{+\infty}$ *such that given prices, the allocation solves the household's problem and markets clear, i.e.* $c_t \leq$ *dividend of the tree,* $s_t = 1$ *and* $B_{jt} = 0$.

After imposing market clearing, the first order conditions with respect to B_{jt} become :

$$q_{1t} = E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} \right) \tag{105}$$

$$q_{jt} = E_t \left(\beta \frac{u'(c_{t+1})}{u'(c_t)} q_{j-1,t+1} \right) \quad j = 2 \ldots J. \tag{106}$$

Iterating forward over the second equation and applying the law of iterated expectation gives the familiar :

$$q_{jt} = E_t \left(\beta^j \frac{u'(c_{t+j})}{u'(c_t)} \right). \tag{107}$$

b. Before computing R_{jt} we note that the log of consumption is an AR(1) process. Namely, we have :

$$\log(c_{t+1}) - \frac{\log(c^*)}{1-\phi} = \phi \left(\log(c_t) - \frac{\log(c^*)}{1-\phi} \right) + \log(\varepsilon_{t+1}).$$

Iterating on this equation, it is then easy to show that :

$$\log(c_{t+j}) - \frac{\log(c^*)}{1-\phi} = \phi^j \left(\log(c_t) - \frac{\log(c^*)}{1-\phi} \right) + \sum_{k=0}^{j-1} \phi^k \log(\varepsilon_{t+j-k}).$$

Consumption growth between period t and $t+j$ is :

$$\log(c_{t+j}) - \log(c_t) = (\phi^j - 1) \log(c_t) + \frac{1-\phi^j}{1-\phi} \log(c^*) + \sum_{k=0}^{j-1} \phi^k \log(\varepsilon_{t+j-k}).$$

Now observe that :

$$
\begin{aligned}
(108) \qquad q_{jt} &= \beta^j E_t\left[\left(\frac{c_{t+j}}{c_t}\right)^{-\theta}\right] \\
(109) \qquad q_{jt} &= \beta^j E_t\left[\exp\left(-\theta(\log(c_{t+j}) - \log(c_t)\right)\right] \\
(110) \qquad q_{jt} &= \beta^j \exp\left(-\theta(\phi^j - 1)\log(c_t) - \theta\frac{1-\phi^j}{1-\phi}\log(c^*)\right) \\
(111) \qquad & \times E_t\left[\exp\left(\sum_{k=0}^{j-1} -\theta\phi^k \log(\varepsilon_{t+j-k})\right)\right].
\end{aligned}
$$

Remember that $\log(\varepsilon_t)$ is normal, with mean 1 and variance σ^2. In particular the $-\theta\phi^k \log(\varepsilon_{t+j-k})$ is normal, with mean $-\theta\phi^k$ and variance $\theta^2\phi^{2k}\sigma^2$. The expectation of $\exp\left((\phi^k \log(\varepsilon_{t+j-k})\right)$ is thus $-\theta\phi^k + \theta^2\phi^{2k}\sigma^2/2$. This gives :

$$
\begin{aligned}
q_{jt} &= \beta^j \exp\left(-\theta(\phi^j - 1)\log(c_t) + -\theta\tfrac{1-\phi^j}{1-\phi}\log(c^*)\right) \\
& \times \exp\left(-\theta\tfrac{1-\phi^j}{1-\phi} + 1/2\theta^2\tfrac{1-\phi^{2j}}{1-\phi}\right).
\end{aligned}
$$

Using the definition $R_{jt} = (1/q_{jt})^{1/j}$, we obtain:

$$
\log(R_{jt}) = a(j) + b(j)\log(c_t). \qquad (112)
$$

Where $b(j) = -\theta(1-\phi^j)/j$ and the constant $a(j)$ collects all the terms that do not depend on c_t. Importantly, $b(j)$ is negative and $|b(j)|$ decreases with maturity [2]. Equation (112) together with these two observations are the basis for answering all the following questions.

c.,d.,e.,f. Since the magnitude of $b(j)$ is decreasing with maturity, if follows that the variance of the interest rates is decreasing with maturity. Similarly, long term interest rates are less responsive than short term interest rates.

Economist C could not make his claim from studying our model. It is true that interest rates are countercyclical because $b(j)$ is negative. However, in our model, causation runs from consumption towards interest rates, not the converse. Specifically, properties of the exogenous consumption process pin down properties of the interest rates.

Lastly, if consumption is very persistent, it is easy to show that the term structure is flat and $b(j) = 0$. An intuition for this result is as follows : the interest rate reflects information about the growth rate of future consumption $\log(c_{t+j}) - \log(c_t)$. When $|\phi| < 1$, the log-consumption process is reverting to its long run value $\log(c^*)/(1-\phi)$. The current consumption level has thus some predictive content about consumption growth rate over subsequent periods: you know it is likely to revert to its mean. When $\phi = 1$ then the log-consumption process is a random walk (with drift). Therefore, the current consumption level has no

[2] to show this fact, study the function $1/x(1-\phi^x)$.

predictive content about future consumption growth rate. Interest rate should not depend on the current consumption level.