Problem Set 4

Problem 1. Computations 1

Consider the one-sector growth model and suppose that $u(c,\ell) = \log c$ and that $F(k,n) = k^{\alpha}n^{1-\alpha}$. Write a code in MATLAB to solve the one-sector growth model using value function iterations. Assume that $\alpha = 1/3$, $\delta = 0.08$, and $\beta = 0.96$. Using your solution calculate the steady state level of capital. Using your computations and by referring to the notes, calculate the rate of convergence of this economy as in Barro and Sala-i-Martin's exercise.

Problem 2. Computations 2

Consider the stochastic one-sector growth model and suppose that $u(c,\ell) = \log c + \psi \log \ell$ and that $y_t = A_t k_t^{\alpha} n_t^{1-\alpha}$. Assume that $\alpha = 1/3$, $\beta = 0.96$ and $\delta = 0.08$.

a. If we were to calibrate ψ so that households on average work, 1/3 of their time, what value do you get for ψ by matching this statistic in the non-stochastic steady state of the model.

b. Suppose that A_t follows a Markov process which is described as follows:

$$A_t \in \{A_L = 0.9, A_H = 1.1\}$$

with the transition matrix given by

$$P = \begin{bmatrix} P_{HH} & P_{HL} \\ P_{LH} & P_{LL} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

where $P_{ij} = \Pr(A_{t+1} = A_i | A_t = A_j)$. Solve the model using value function iteration in MATLAB.

c. Calculate the stationary distribution associated with the policy functions that you found in part b.

d. Using the stationary distribution that you found in part c, calculate standard deviation of GDP, consumption, investment, and hours. Discuss the results.

Problem 3. The Merton Problem

Consider a consumer who has standard CRRA utility function

$$\sum_{t=0}^{T} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

and lives for T periods. Suppose that in each period the consumer can invest in a risk-free asset with gross return R_f and a risky assets whose gross return is given by R_t which is an i.i.d. random variable distributed according to some continuous distribution $F(R)$. Suppose that initially, the consumer's wealth is given by W_0.

a. Write down the sequence of budget constraints for the consumer.

Solution. We have

$$
\begin{aligned}
c_t + b_t + a_t &= W_t \\
W_{t+1} &= R_f b_t + R_t a_t
\end{aligned}
$$

where a_t is holding of the risky asset and b_t is the holding of the safe asset.

b. Formulate the consumer's problem recursively.

Solution. Consumer's problem in recursive form is given by

$$V_t\left(W\right) = \max \frac{c^{1-\sigma}}{1-\sigma} + \beta \int V_{t+1}\left(Ra + R_f b\right) dF\left(R\right)$$

subject to

$$c + a + b = W$$

c. Solve the consumer's problem using a guess and verify method.

Solution. We can guess that

$$V_{t+1}\left(W\right) = B_{t+1} \frac{W^{1-\sigma}}{1-\sigma}$$

Then the optimization is given by

$$\max_{a,b} \frac{\left(W-a-b\right)^{1-\sigma}}{1-\sigma} + \beta B_{t+1} \int \frac{\left(Ra + R_f b\right)^{1-\sigma}}{1-\sigma} dF\left(R\right) \tag{1}$$

The FOC associated with this is given by

$$
\begin{aligned}
\left(W-a-b\right)^{-\sigma} &= \beta B_{t+1} \int R\left(Ra + R_f b\right)^{-\sigma} dF\left(R\right) \\
\left(W-a-b\right)^{-\sigma} &= \beta B_{t+1} R_f \int \left(Ra + R_f b\right)^{-\sigma} dF\left(R\right)
\end{aligned}
$$

If we define $\hat{a} = a/W$ and $\hat{b} = b/W$, then

$$
\begin{aligned}
\left(1-\hat{a}-\hat{b}\right)^{-\sigma} &= \beta B_{t+1} \int R\left(R\hat{a} + R_f \hat{b}\right)^{-\sigma} dF\left(R\right) \\
\left(1-\hat{a}-\hat{b}\right)^{-\sigma} &= \beta B_{t+1} R_f \int \left(R\hat{a} + R_f \hat{b}\right)^{-\sigma} dF\left(R\right)
\end{aligned}
$$

This a system of two equations and two unknowns and can be shown to have a unique solution. To see this, note that we have

$$\int \left(R - R_f\right)\left(R\hat{a} + R_f \hat{b}\right)^{-\sigma} dF\left(R\right) = 0$$

Dividing the above by $\hat{a}^{-\sigma}$ and defining $x = \hat{b}/\hat{a}$, we must have

$$\int (R - R_f)(R + xR_f)^{-\sigma} dF(R) = 0 \tag{2}$$

The derivative of the LHS of the above w.r.t. x is

$$-\sigma \int (R - R_f) R_f (R + xR_f)^{-\sigma-1} dF(R) = \sigma (R_f - \mathbb{E}R) \mathbb{E}(R + xR_f)^{-1-\sigma} - \sigma \mathbb{E}\left[(R - \mathbb{E}R)(R + xR_f)^{-1-}\right.$$

$$= \sigma (R_f - \mathbb{E}R) \mathbb{E}(Rx + R_f)^{-1-\sigma} - \sigma Cov\left(R, (R + xR_f)^{-1-\sigma}\right)$$

If $R_f \geq \mathbb{E}R$, then the above is a positive expression since $Cov\left(R, (Rx + R_f)^{-1-\sigma}\right)$ and the first term is a positive expression. This means that the above has a unique solution for x.

We then have

$$(1 - \hat{a}(1 + x))^{-\sigma} = \beta \hat{a}^{-\sigma} B_{t+1} \int R(R + R_f x)^{-\sigma} dF(R)$$

The LHS of the above is increasing in $\hat{a}$ and the RHS is decreasing in $\hat{a}$. In addition, the RHS is is ∞ at $\hat{a} = 0$ and the LHS is ∞ at $\hat{a} = \frac{1}{1+x}$. Thus, the solution to the above is positive and unique. If we let $\hat{a}_{t+1}$ and $\hat{b}_{t+1}$ be the solution to the above system of equations, the objective in 1 is given by

$$W^{1-\sigma} \frac{\left(1 - \hat{a}_{t+1} - \hat{b}_{t+1}\right)^{1-\sigma}}{1-\sigma} + \beta B_{t+1} \int \frac{W^{1-\sigma}}{1-\sigma} \left(R\hat{a}_{t+1} + R_f \hat{b}_{t+1}\right)^{1-\sigma} dF(R) =$$

$$\frac{W^{1-\sigma}}{1-\sigma} \left[\left(1 - \hat{a}_{t+1} - \hat{b}_{t+1}\right)^{1-\sigma} + \beta B_{t+1} \int \left(R\hat{a}_{t+1} + R_f \hat{b}_{t+1}\right)^{1-\sigma} dF(R)\right] = B_t \frac{W^{1-\sigma}}{1-\sigma}$$

which verifies our guess.

d. From part c, what does the portfolio choice of the consumer look like, i.e., what fraction of his/her investments in each period are in bonds vs stocks? How does this depend on age/time? Is this in line with the advice given by financial planners? If not, what ingredients can be added to the model to address this discrepancy?

Solution. From the above, we have that the ratio of bond holdings to stock holdings is given by

$$\frac{b_t}{a_t} = x$$

where x satisfies (2) which is independent of age. Typically financial planners and pension funds recommend allocating a higher fraction of portfolio to be allocated to bonds, which is completely the opposite of what the model prescribes. One possibility is that later in life, a decline in labor income effectively makes the individual more risk-averse and thus they must allocation more of their portfolio to safe assets. The problem with this approach is that labor income is also correlated with the stock market which makes young and working age population wanting to hold more bonds. Another argument for the advice is that there are various expense shocks that hit individuals late in life and make them effectively more risk-averse. See the paper by Canner, Mankiw and Weil (1994) for discussions of possible explanations.

Problem 4.

Solve problems 14.2, 14.3, 14.4, 14.8, 14.9, 14.10.