CHAPTER 5

Search, matching, and unemployment

Exercise 5.1. *Being unemployed with only a chance of an offer*

An unemployed worker samples wage offers on the following terms. Each period, with probability ϕ, $1 > \phi > 0$, she receives no offer (we may regard this as a wage offer of zero forever). With probability $(1-\phi)$ she receives an offer to work for w forever, where w is drawn from a cumulative distribution function $F(w)$. Successive drawings across periods are independently and identically distributed. The worker chooses a strategy to maximize

$$E\sum_{t=0}^{\infty}\beta^t y_t, \qquad \text{where } 0<\beta<1,$$

$y_t = w$ is the worker is employed, and $y_t = c$ is the worker is unemployed. Here c is unemployment compensation, and w is the wage at which the worker is employed. Assume that, having once accepted a job offer at wage w, the worker stays in the job forever.
Let $v(w)$ be the expected value of $\sum_{t=0}^{\infty}\beta^t y_t$ for an unemployed worker who has offer w in hand and who behaves optimally. Write Bellman's functional equation for the worker's problem.

Solution

Let $v(w)$ be the expected value of $\sum_{t=0}^{\infty}\beta^t y_t$ for an unemployed worker who has offer w in hand and who behaves optimally.

$$v(w) = \max_{A,R}\left\{\frac{w}{1-\beta}, c+\phi\beta v(0)+(1-\phi)\beta\int v(w')dF(w')\right\}. \tag{27}$$

Here the maximization is over the two actions: accept the offer to work forever at wage w, or reject the current offer and take a chance on drawing a new offer next period.

Exercise 5.2. *Two offers per period*

Consider an unemployed worker who each period can draw *two* independently and identically distributed wage offers from the cumulative probability distribution function $F(w)$. The worker will work forever at the same wage after having once accepted an offer. In the event of unemployment during a period, the worker receives unemployment compensation c. The worker derives a decision rule to maximize $E\sum_{t=0}^{\infty}\beta^t y_t$, where $y_t = w$ or $y_t = c$, depending on whether she is employed or unemployed. Let $v(w)$ be the value of $E\sum_{t=0}^{\infty}\beta^t y_t$ for a currently unemployed worker who has best offer w in hand.

a. Formulate Bellman's equation for the worker's problem.

b. Prove that the worker's reservation wage is *higher* than it would be had the worker faced the same c and been drawing only *one* offer from the same distribution $F(w)$ each period.

Solution

a. Note that the event $\max\{w_1, w_2\} < w$ is the event $(w_1 < w) \cap (w_2 < w)$. Therefore $\text{prob}\{\max(w_1, w_2) < w\} = F(w)^2$. The worker will evidently limit his choice to the larger of the two offers each period. Bellman's equation is therefore

$$v(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \int v(w')d(F^2)(w')\right\},$$

where w is the best offer in hand.

b. The reservation wage obeys the following equation:

$$(\bar{w}_2 - c) = \frac{\beta}{1-\beta}\int_{\bar{w}_2}^{\infty}(w' - \bar{w}_2)d(F^2)(w').$$

Using the usual integration by part argument, one obtains the equation:

$$h_2(\bar{w}_2) \equiv (1-\beta)\bar{w}_2 - \beta\int_{\bar{w}_2}^{B}(1 - F(w')^2)dw' = 0.$$

Observe that h_2 is an increasing function. When the worker is given only one offer, the reservation wage solves :

$$h_1(\bar{w}_1) \equiv (1-\beta)\bar{w}_1 - \beta\int_{\bar{w}_1}^{B}(1 - F(w'))dw' = 0.$$

Since $F(w)^2 \leq F(w)$, we have $h_2(w) \leq h_1(w)$. Therefore:

$$0 = h_1(\bar{w}_1) = h_2(\bar{w}_2) \leq h_1(\bar{w}_2).$$

Since h_2 is increasing if follows that

$$\bar{w}_1 \leq \bar{w}_2.$$

The intuition underlying this result is as follows: the worker could choose always to ignore the second offer. This policy, possibly suboptimal, would leave the worker with a decision problem that is formally identical to the standard one-offer problem. The value of the objective function of the true problem is at least as high as the value of the objective function under the artificially restricted problem. Because the reservation wage has the property of equating the value of accepting a job, $w/(1-\beta)$, with the value of rejecting, $c + \beta Ev(w')$, a higher value of $Ev(w')$, which results in the two-offer case, requires a higher reservation wage.

Exercise 5.3. *A random number of offers per period*

An unemployed worker is confronted with a random number, n, of job offers each period. With probability π_n, the worker receives n offers in a given period, where $\pi_n \geq 0$ for $n \geq 1$, and $\sum_{n=1}^{\infty}\pi_n = 1$ for $N < +\infty$. Each offer is drawn independently from the same distribution $F(w)$. Assume that the number of offers n is independently distributed across time. The worker works forever at wage w after having accepted a job and receives unemployment compensation

of c during each period of unemployment. He chooses a strategy to maximize $E\sum_{t=0}^{\infty}\beta^t y_t$ where $y_t = c$ if he is unemployed, $y_t = w$ if he is employed.
Let $v(w)$ be the value of the objective function of an unemployed worker who has best offer w in hand and who proceeds optimally. Formulate Bellman's equation for this worker.

Solution

$$v(w) = \max\left\{\frac{w}{1-\beta}, c + \sum_{n=1}^{N}\pi_n \int v(w')d(F^n)(w')\right\}.$$

In effect, the worker is confronted with a lottery with probabilities π_n over distributions $F^n(w)$, from which he will sample next period. As in Exercise 2.1, w is the highest offer in hand.

Exercise 5.4. *Cyclical fluctuations in number of job offers*

Modify Exercise 5.3 as follows. Let the number of job offers n follow a Markov process, with

$$\begin{aligned}&\text{prob}\,\{\text{number of offers next period} = m | \text{number of offers this}\\ &\quad\text{period} = n\} = \pi_{mn}, \qquad m = 1,\dots,N, \quad n = 1,\dots,N\\ &\textstyle\sum_{m=1}^{N}\pi_{mn} = 1 \qquad \text{for} \quad n = 1,\dots,N.\end{aligned} \tag{28}$$

Here $[\pi_{mn}]$ is a "stochastic matrix" generating a Markov chain. Keep all other features of the problem as in Exercise 2.3. The worker gets n offers per period, where n is now generated by a Markov chain so that thenumber of offers is possibly correlated over time.

a. Let $v(w,n)$ be the value of $E\sum_{t=0}^{\infty}\beta^t y_t$ for an unemployed worker who has received n offers this period, the best of which is w. Formulate Bellman's equation for the worker's problem.

b. Show that the optimal policy is to set a reservation wage $\bar{w}(n)$ that depends on the number of offers received this period.

Solution

a. The Bellman equation for the worker's problem is

$$v(w,n) = \max_{\text{accept,reject}}\left\{\frac{w}{1-\beta}, c + \sum_{m=1}^{N}\pi_{m,n}\int v(w',m)d(F^m)(w')\right\}. \tag{29}$$

b. From equation (29), we see that the right branch of the right side of the functional equation is evidently a function only of n. The argument in the text applies for each n and implies a reservation wage that is a function of n.

Exercise 5.5. *Choosing the number of offers*

An unemployed worker must choose the number of offers n to solicit. At a cost of $k(n)$ the worker receives n offers this period. Here $k(n+1) > k(n)$ for $n \geq 1$. The number of offers n must be chosen in advance at the beginning of the period and cannot be revised during the period. The worker wants to maximize $E \sum_{t=0}^{\infty} \beta^t y_t$. Here y_t consists of w each period she is employed but not searching, $[w - k(n)]$ the first period she is employed but searches for n offers, and $[c - k(n)]$ each period she is unemployed but solicits and rejects n offers. The offers are each independently drawn from $F(w)$. The worker who accepts an offer works forever at wage w.
Let Q be the value of the problem for an unemployed worker who has not yet chosen the number of offers to solicit. Formulate Bellman's equation for this worker.

Solution

$$Q = \max_n \int \max_{\substack{\text{accept}\\ \text{reject}}} \left\{ \frac{w}{1-b} - k(n), -k(n) + \beta Q \right\} d(F^n)(w).$$

The worker proceeds sequentially each period, first choosing n, then deciding whether to accept or reject the best offer.

Exercise 5.6. *Mortensen externality*

Two parties to a match (say, worker and firm) jointly draw a match parameter θ from a c.d.f. $F(\theta)$. Once matched, they stay matched forever, each one deriving a benefit of θ per period from the match. Each unmatched pair of agents can influence the number of offers received in a period in the following way. The worker receives n offers per period, with $n = f(c_1 + c_2)$, where c_1 is the resources the worker devotes to searching and c_2 is the resources the typical firm devotes to searching. Symmetrically, the representative firm receives n offers per period where $n = f(c_1 + c_2)$. (We shall define the situation so that firms and workers have the same reservation θ so that there is never unrequited love.) Both c_1 and c_2 must be chosen at the beginning of the period, prior to searching during the period. Firms and workers have the same preferences, given by the expected present value of the match parameter θ, net of search costs. The discount factor β is the same for worker and firm.

a. Consider a Nash equilibrium in which party i chooses c_i, taking c_j, $j \neq i$, as given. Let Q_i be the value for an unmatched agent of type i before the level of c_i has been chosen. Formulate Bellman's equation for agents of type 1 and 2.

b. Consider the social planning problem of choosing c_1 and c_2 sequentially so as to maximize the criterion of λ times the utility of agent 1 plus $(1 - \lambda)$ times the utility of agent 2, $0 < \lambda < 1$. Let $Q(\lambda)$ be the value for this problem for two unmatched agents before c_1 and c_2 have been chosen. Formulate Bellman's equation for this problem.

c. Comparing the results in (a) and (b), argue that, in the Nash equilibrium, the optimal amount of resources has not been devoted to search.

Solution

a.

$$Q_1 = \max_{c_1} \int \max_{\text{accept,reject}} \left\{ \frac{\theta}{1-\beta} - c_1, -c_1 + \beta Q_1 d(F^n)(\theta) \right\},$$

subject to $n = f(c_1 + c_2)$, c_2 given

$$Q_2 = \max_{c_2} \int \max_{\text{accept,reject}} \left\{ \frac{\theta}{1-\beta} - c_2, -c_2 + \beta Q_2 \right\} d(F^n)(\theta)$$

subject to $n = f(c_1 + c_2)$, c_1 given.

b.

$$\begin{aligned} Q(\lambda) = \quad & \max_{c_1,c_2} \Big\{ \int \max_{\text{accept,reject}} \Big\{ \lambda \tfrac{\theta}{1-\beta} - \lambda c_1 + (1-\lambda)\Big(\tfrac{\theta}{1-\beta} - c_2\Big), \\ & -\lambda c_1 - (1-\lambda) c_2 + \beta Q(\lambda) \Big\} d(F^n)(\theta) \Big\} \end{aligned}$$

subject to $n = f(c_1 + c_2)$.

c. The Nash equilibrium is a (c_1, c_2) pair that solves the two functional equations in (a). In general, this (c_1, c_2) pair will not solve the functional equation in (b) because each agent in (a) neglects the effects of his choice of c_j on the welfare of the other agent. In general, there will be too little search in the Nash equilibrium if $f(c_1 + c_2)$ is increasing in $(c_1 + c_2)$.

Exercise 5.7. *Variable labor supply*

An unemployed worker receives each period a wage offer w drawn from the distribution $F(w)$. The worker has to choose whether to accept the job – and therefore to work forever – or to search for another offer and collect c in unemployment compensation. The worker who decides to accept the job must choose the number of hours to work in each period. The worker chooses a strategy to maximize

$$E \sum_{t=0}^{\infty} \beta^t u(y_t, l_t), \qquad \text{where } 0 < \beta < 1,$$

and $y_t = c$ if the worker is unemployed, and $y_t = w(1 - l_t)$ if the worker is employed and works $(1 - l_t)$ hours; l_t is leisure with $0 \leq l_t \leq 1$.
Analyze the worker's problem. Argue that the optimal strategy has the reservation wage property. Show that the number of hours worked is the same in every period.

Solution

Let s be the state variable. We choose $s = (w, 0)$, where w is the wage offer and $0 = E$ if the worker is employed, and $0 = U$ if she is unemployed. Consider first the situation of an employed worker. Bellman's equation is

$$v(w, E) = \max_{l} \{ u[w(1-l), l] + \beta v(w, E) \}.$$

Then it follows that

$$v(w,E) = \frac{u\left(w(1-l(w)), l(w)\right)}{1-\beta},$$

where $l(w) \equiv \underset{l}{\text{argmax}}\ u(w(1-l), l)$.

Let's show that $v(w,E)$ is increasing in w. Consider $w_1 < w_2$. We have :

$$\begin{aligned} u\left(w_1(1-l(w_1)), l(w_1)\right) &\leq u\left(w_2(1-l(w_1)), l(w_1)\right) \\ &\leq \max_l u\left(w_2(1-l), l\right) \\ &\equiv u\left(w_2(1-l(w_2)), l(w_2)\right). \end{aligned}$$

Intuitively, a worker receiving $w_2 > w_1$ has the option work $1 - l(w_1)$ hours paid w_2, that yields a higher utility than working $1 - l(w_1)$ hours paid w_1. Its optimal choice $1 - l(w_2)$ necessarily yields an even higher utility.

Now consider an unemployed worker. Bellman's equation is

$$v(w,U) = \max_{\text{accept,reject}} \left\{ V(w,E), u(c,1) + \beta \int v(w',U)dF(w') \right\}.$$

The outside maximization is over two actions: accept the offer (in which case the worker chooses l optimally) or reject the offer, collect unemployment compensation, and wait for a new offer next period. The first term is incresing in w and the second is independent of w. Therefore the optimal policy is to accept offers offers that are at least equal to some $\bar{w}$. Once an offer has been accepted, hours worked are constant and equal to $l(w)$.

Exercise 5.8. *Wage growth rate and the reservation wage*

An unemployed worker receives each period an offer to work for wage w_t forever, where $w_t = w$ in the first period and $w_t = \phi^t w$ after t periods in the job. Assume $\phi > 1$, that is, wages increase with tenure. The initial wage offer is drawn from a distribution $F(w)$ that is constant over time (entry-level wages are stationary); successive drawings across periods are independently and identically distributed. The worker's objective function is to maximize

$$E \sum_{t=0}^{\infty} \beta^t y_t, \qquad \text{where } 0 < \beta < 1,$$

and $y_t = w_t$ if the worker is employed and $y_t = c$ if the worker is unemployed, where c is unemployment compensation. Let $v(w)$ be the optimal value of the objective function for an unemployed worker who has offer w in hand. Write Bellman's equation for this problem. Argue that, if two economies differ only in the growth rate of wages of employed workers, say $\phi_1 > \phi_2$, the economy with the higher growth rate has the smaller reservation wage.

Note. Assume that $\phi_i\beta < 1,\ i = 1, 2$.

Solution

If the worker accepts employment at wage w, the sequence $\{y_t\}$ is given by $y_t = w,\ y_{t+1} = \phi w \ldots, y_{t+j} = \phi^j w \ldots$. Therefore the value of the objective function if the worker accepts is $\sum_{j=0}^{\infty} \beta^j y_{t+j} = w/(1-\beta\phi)$. Bellman's equation for the worker's problem is

$$v(w) = \max\left\{\frac{w}{1-\beta\phi}, c + \beta \int v(w')dF(w')\right\}.$$

Using the same argument as when studying McCall's model, one shows that that the optimal policy is to accept all offers to work with an initial wage higher than a reservation wage $\bar{w}$.

$$v(w) = \begin{cases} \frac{\bar{w}}{1-\phi\beta} & w \leq \bar{w} \\ \frac{w}{1-\phi\beta} & w \geq \bar{w}. \end{cases}$$

Because, at $w = \bar{w}$, we have

$$\frac{\bar{w}}{1-\phi\beta} = c + \beta \int_0^B v(w')dF(w'),$$

we get, after substituting for $v(w)$ its expression,

$$\frac{\bar{w}}{(1-\phi\beta)} = \frac{c+\beta}{1-\phi\beta}\bar{w} \int_0^{\bar{w}} dF(w') + \frac{\beta}{1-\phi\beta} \int_{\bar{w}}^B w' dF(w').$$

This equation can be rearranged to give

$$(1-\beta)\bar{w} - \beta \int_{\bar{w}}^B (w' - \bar{w})dF(w') = (1-\beta\phi)c.$$

It is easy to see (using the Leibniz rule) that the left-hand side is increasing in $\bar{w}$. Therefore, if $\phi_1 > \phi_2$, that is, $(1-\beta\phi_1)c < (1-\beta\phi_2)c$, it must be that $\bar{w}_1 < \bar{w}_2$. The intuition behind this result is simple: for any given offer w, the value of accepting the offer is higher, the higher the growth rate of wages ϕ. Therefore, the sooner an offer is accepted, the sooner the benefits of the growth in wages are realized. This pattern makes some job offers more attractive even though the initial wage is not very high.

Exercise 5.9. *Search with a finite horizon*

Consider a worker who lives two periods. In each period the worker, if unemployed, receives an offer of lifetime work at wage w, where w is drawn from a distribution F. Wage offers are identically and independently distributed over time. The worker's objective is to maximize $E\{y_1 + \beta y_2\}$, where $y_t = w$ if the worker is employed and is equal to c – unemployment compensation – if the worker is not employed.

Analyze the worker's optimal decision rule. In particular, establish that the optimal strategy is to choose a reservation wage in each period and to accept any offer with a wage at least as high as the reservation wage and to reject offers below that level. Show that the reservation wage decreases over time.

Solution

We first analyze the worker's problem in the second period of life. We consider an unemployed worker; an employed worker does not have to solve any decision problem. Let $v_2(w)$ be the optimal value of the problem for an unemployed worker with offer w in hand. Then $v_2(w) = \max\{w, c\}$. It follows that the optimal strategy is to accept offers that are at least c and to reject all others. The second-period reservation wage, $\bar{w}_2$, is equal to c. In the first period if the worker is faced with a wage w and accepts the offer, the value of the objective function is $w(1+\beta)$. If the worker rejects he gets c in the first period and $v_2(w')$, a random variable, in the following period. The expected value of rejecting the offer is thus $c + \beta \int_0^\infty v_2(w')dF(w')$.
Therefore the optimal value of the objective function for a worker with offer w in hand is given by

$$v_1(w) = \max\left\{w(1+\beta), c + \beta \int_0^B v_2(w')dF(w')\right\}.$$

Notice that the second term in brackets is constant, whereas the first is increasing in w. It follows that the optimal policy is of the reservation wage form. There exists a $\bar{w}_1$ such that, for $w \leq \bar{w}_1$, the second term is higher, and therefore the optimal strategy is to reject the job offer and to remain unemployed. Similarly, when $w > \bar{w}_1$, the first term is higher and the optimal strategy is to accept the job. As usual $\bar{w}_1$ satisfies :

$$w_1(1+\beta) = c + \beta \int_0^B v_2(w')dF(w').$$

Observe that $v_2(w) = \max\{w, c\} \geq c$. In words, a worker who is unemployed in the second period get at least c, the unemployement compensation. Thus $\int_0^B v_2(w')dF(w') = E(v_2(w')) \geq c$, with a strict inequality if $\Pr(w' \geq c) > 0$. This inequality implies :

$$w_1(1+\beta) = c + \beta \int_0^B v_2(w')dF(w') \geq (1+\beta)c.$$

The reservation wage decreases as the retirement date approaches. The intuition underlying this result is that, the shorter the horizon, the smaller the benefits of "waiting to see if next period the wage offer is really high" (the option value of waiting) because those benefits cannot be enjoyed for a long period. The implication is hence that the alternative to waiting – that is, accepting a job – becomes more attractive. This aspect is reflected in the model by a decrease in the reservation wage, which in fact corresponds to an increase in the percentage of job offers that are accepted.

Exercise 5.10. *Finite horizon and mean-preserving spread*

Consider a worker who draws every period a job offer to work forever at wage w. Successive offers are independently and identically distributed drawings from a distribution $F_i(w),\ i = 1, 2$. Assume that F_1 has been obtained from F_2 by a mean-preserving spread (see Section 2.4). The worker's objective is to maximize

$$E \sum_{t=0}^{T} \beta^t y_t, \qquad 0 < \beta < 1,$$

where $y_t = w$ is the worker has accepted employment at wage w and is zero otherwise. Assume that both distributions, F_1 and F_2, share a common upper bound, B.

a. Show that the reservation wages of workers drawing from F_1 and F_2 coincide at $t = T$ and $t = T - 1$.

b. Argue that for $t \leq T - 2$ the reservation wage of the workers that sample wage offers from the distribution F_1 is higher than the reservation wage of the workers that sample from F_2. **c.** Now introduce unemployment compensation: the worker who is unemployed collects c dollars. Prove that the result in (a) no longer holds, that is, the reservation wage of the workers that sample from F_1 is higher than the one corresponding to workers that sample from F_2 for $t = T - 1$.

Solution

a. Let $v_t^i(w)$ be the optimal value of the objective function of an unemployed worker at time t who has offer w in hand and draws wage offers from the distribution $F_i,\ i = 1, 2$. Then it is clear that $v_T^i(w) = \max\{0, w\} = w$. Therefore $\int_0^B v_T^i(w) dF_i(w) = \int_0^B w dF_i(w) = Ew,\ i = 1, 2$. Clearly the reservation wage at time T is zero: the worker accepts every offer. At time $(T - 1)$, Bellmans' equation for the worker's problem is

$$\begin{aligned} v_{T-1}^i(w) &= \max\left\{w(1+\beta), \beta \int_0^B v_T^i(w') F_i(dw')\right\} \\ &\max\left\{w(1+\beta), \beta Ew\right\}. \end{aligned}$$

It is then clear that the worker will accept the offer if $w(1 + \beta) \geq \beta Ew$ and will reject it otherwise. Therefore the reservation wage $\bar{w}_{T-1}$ is $\beta Ew/(1+\beta)$. Because the expectation of w is the same no matter whether w is drawn from F_1 or F_2, it follows that both types of workers have the same reservation wage.

b. We prove this point by induction. Assume that at $t+1$ the optimal policy under both distribution is of the reservation wage form. Also, assume that $w_{t+1}(1)$, the reservation wage under c.d.f. F_1, is greater than $w_{t+1}(2)$, the reservation wage under c.d.f. F_2. Observe that those two assumptions are true at time T. The Bellman equation at time t is:

$$v_t^i(w) = \max\left\{w \frac{1 - \beta^{T-t+1}}{1 - \beta}, \beta \int_0^B v_{t+1}^i(w') dF_i(w')\right\},$$

where $w\frac{1-\beta^{T-t+1}}{1-\beta}$ is the value of working at wage w in periods $t, t+1, \ldots T$. The first term is increasing in w while the second one is constant. It follows that, at time t, the optimal policy is also of the reservation wage form. Furthermore, the time t reservation wage $w_t(i)$ solves the usual indifference condition:

$$\begin{aligned}
w_t(i)\tfrac{1-\beta^{T-t+1}}{1-\beta} &= \beta \textstyle\int_0^B v_{t+1}^i(w')dF_i(w') \\
w_t(i)\tfrac{1-\beta^{T-t+1}}{1-\beta} &= \beta \textstyle\int_0^{w_{t+1}(i)} \tfrac{1-\beta^{T-t}}{1-\beta} w_{t+1}(i)dF_i(w') + \beta \int_{w_{t+1}(i)}^B \tfrac{1-\beta^{T-t}}{1-\beta} w' dF_i(w') \\
w_t(i)\tfrac{1-\beta^{T-t+1}}{1-\beta} &= \beta \tfrac{1-\beta^{T-t}}{1-\beta}\left(\textstyle\int_0^{w_{t+1}(i)} \left(w_{t+1}(i) - w'\right) dF_i(w') + \int_0^B w' dF_i(w')\right).
\end{aligned}$$

Integrating the first term by part and rearanging yields :

$$w_t(i) = \frac{\beta - \beta^{T-t+1}}{1-\beta^{T-t+1}}\left(\int_0^{w_{t+1}(i)} F_i(w')dw' + E_i(w)\right).$$

Observe that $E_1(w) = E_2(w)$ by assumption. Also, by definition of a mean preserving spread and since $w_{t+1}(1) \geq w_{t+1}(2)$, we have:

$$\begin{aligned}
\textstyle\int_0^{w_{t+1}(1)} F_1(w')dw' &\geq \textstyle\int_0^{w_{t+1}(2)} F_1(w')dw' + \int_{w_{t+1}(2)}^{w_{t+1}(1)} F_2(w')dw' \\
&\geq \textstyle\int_0^{w_{t+1}(2)} F_2(w')dw' + \int_{w_{t+1}(2)}^{w_{t+1}(1)} F_2(w')dw'.
\end{aligned}$$

Therefore $w_t(1) \geq w_t(2)$.

c. The value of the problem at $t = T$ is $v_T^i(w) = \max\{w, c\},\ i = 1, 2$. Then $\bar{w}_T^1 = \bar{w}_T^2 = c$. If we use the same argument as in (b), however, it follows directly that $\int_0^B \max\{w, c\}dF_1(w) \geq \int_0^B \max\{w, c\}dF_2(w)$, or $Ev_T^1 \geq Ev_T^2$. On the other hand, the reservation wage at $(T-1)$ satisfies $\bar{w}_{T-1}^i = \beta/(1+\beta)Ev_T^i$. Therefore $\bar{w}_{T-1}^1 \geq \bar{w}_{T-1}^2$.

Exercise 5.11. *Pissarides' Analysis of Taxation and Variable Search Intensity*

An unemployed worker receives each period a zero offer (or no offer) with probability $[1-\pi(e)]$. With probability $\pi(e)$ the worker draws an offer w from the distribution F. Here e stands for effort – a measure of search intensity – and $\pi(e)$ is increasing in e. A worker who accepts a job offer can be fired with probability α, $0 < \alpha < 1$. The worker chooses a strategy, that is, whether to accept an offer or not and how much effort to put into search when unemployed, to maximize

$$E\sum_{t=0}^{\infty} \beta^t y_t, \qquad 0 < \beta < 1,$$

where $y_t = w$ if the worker is employed with wage w and $y_t = 1 - e + z$ if the worker spends e units of leisure searching and does not accept a job. Here z is unemployment compensation. For the worker who searched and accepted a job, $y_t = w - e - T(w)$; that is, in the first period the wage is net of search costs. Throughout, $T(w)$ is the amount paid in taxes when the worker is employed. We assume that $w - T(w)$ is increasing in w. Assume that $w - T(w) = 0$ for $w = 0$, that, if $e = 0$, $\pi(e) = 0$ – that is, the worker gets no offers – and that $\pi'(e) > 0$, $\pi''(e) < 0$.

a. Analyze the worker's problem. Establish that the optimal strategy is to choose a reservation wage. Display the condition that describes the optimal choice of e, and show that the reservation wage is independent of e.

b. Assume that $T(w) = t(w - a)$ where $0 < t < 1$ and $a > 0$. Show that an increase in a decreases the reservation wage and increases the level of effort, increasing the probability of accepting employment.

c. Show under what conditions a change in t has the opposite effect.

Solution

a. Let the state variable that completely summarizes current and future opportunities be $x = (w, e, s)$, where w is the wage, e is the effort, and $s = E$ if the worker is employed and $s = U$ if he is unemployed. Recall that, if the worker is employed, then $e = 0$. Let Q be the expected value of the objective function for an unemployed worker who behaves optimally before getting an offer. Then if the worker is employed, the value of the objective function is given by

$$v(w, 0, E) = w - T(w) + \beta(1 - \alpha)v(w, 0, E) + \beta\alpha Q,$$

or

$$v(w, 0, E) = \frac{w - T(w)}{1 - \beta(1 - \alpha)} + \frac{\beta\alpha Q}{1 - \beta(1 - \alpha)}.$$

If the worker is unemployed, has an offer w in hand, and spent $e > 0$ units of leisure searching this period, the value of the objective function is

$$\begin{aligned} v(w, e, U) = \;& \max\{w - T(w) - e + \beta(1 - \alpha)v(w, 0, E) \\ & + \beta\alpha Q, 1 - e + z + \beta Q\}, \end{aligned}$$

where the first term reflects the value of accepting employment and the second the value of rejecting the offer. Using the expression we found for $v(w, 0, E)$, we get

$$\begin{aligned} v(w, e, U) = \;& \max\left\{\tfrac{w - T(w)}{1 - \beta(1 - \alpha)} - e \right. \\ & \left. + \tfrac{\beta\alpha Q}{1 - \beta(1 - \alpha)}, 1 - e + z + \beta Q\right\}. \end{aligned}$$

Then, using a standard argument, we see from the above equation that the optimal strategy is to accept offers greater than or equal to $\bar{w}$ and to reject all others; $\bar{w}$ is such that it makes the worker indifferent between accepting or rejecting the job offer; that is, $\bar{w}$ solves

$$\frac{\bar{w} - T(\bar{w})}{1 - \beta(1 - \alpha)} - e + \frac{\beta\alpha Q}{1 - \beta(1 - \alpha)} = 1 - e + z + \beta Q,$$

or

$$\bar{w} - T(\bar{w}) = [1 - \beta(1 - \alpha)](1 + z + \beta Q) - \beta\alpha Q. \tag{30}$$

Notice that we cannot use this expression for $\bar{w}$ to compute the reservation wage, because Q must be determined endogenously. It is clear, however, that, if Q is independent of e (as we will show that it is), then $\bar{w}$ does not depend on e.
Because we established that the optimal policy is of the reservation wage variety, we can compute $v(w,e,U)$. This function is given by

$$v(w,e,U) = \begin{cases} \frac{w-T(w)}{1-\beta(1-\alpha)} - e + \frac{\beta\alpha Q}{1-\beta(1-\alpha)} & w \geq \bar{w} \\ 1 - e + z + \beta Q & w \leq \bar{w}. \end{cases}$$

Let $\Phi(e) = Ev(w,e,U) = \int_0^\infty v(w,e,U)F(dw)$,

$$\begin{aligned} \Phi(e) = \; & (1+z+\beta Q)F(\bar{w}) \\ & + \frac{1}{1-\beta(1-\alpha)} \int_{\bar{w}}^\infty [w - T(w)]F(dw) - e \\ & + [1 - F(\bar{w})]\frac{\beta\alpha Q}{1-\beta(1-\alpha)}. \end{aligned}$$

Because we have shown that

$$\frac{\beta\alpha Q}{1-\beta(1-\alpha)} = (1+z+\beta Q) - \frac{\bar{w} - T(\bar{w})}{1-\beta(1-\alpha)},$$

we have, after some substitution, that

$$\begin{aligned} \Phi(e) = \; & \frac{1}{1-\beta(1-\alpha)} \\ & \cdot \int_{\bar{w}}^\infty ([w - T(w)] - [\bar{w} - T(\bar{w})])F(dw) + 1 + z + \beta Q - e. \end{aligned}$$

Now consider $\Phi(0)$. Recall that, if $e = 0$, the worker gets no offers, and hence $v(w,0,U) = 1 + z + \beta Q$. This expression is independent of w, and so $\Phi(0) = 1 + z + \beta Q$. Therefore

$$\begin{aligned} \Phi(e) = \; & \frac{1}{1-\beta(1-\alpha)} \\ & \cdot \int_{\bar{w}}^\infty ([w - T(w)] - [\bar{w} - T(\bar{w})])F(dw) + \Phi(0) - e. \end{aligned}$$

To simplify notation let $(w - T(w)) - (\bar{w} - T(\bar{w})) \equiv \Delta Y(w)$. Then the above expression becomes

$$\Phi(e) = \frac{1}{1-\beta(1-\alpha)} \int_{\bar{w}}^\infty \Delta Y(w)F(dw) + \Phi(0) - e.$$

If the worker chooses to spend e units of effort, he gets an offer with probability $\pi(e)$ and expected value $\Phi(e)$. With probability $[1-\pi(e)]$ he gets no offers. This alternative has value $\Phi(0) - e$.
Then the value of the problem for an unemployed worker who behaves optimally is given by Q, where Q satisfies

$$\begin{aligned} Q \; & \equiv \max_{0 \leq e \leq 1} \{\pi(e)\Phi(e) + [1-\pi(e)][\Phi(0) - e]\} \\ Q \; & \equiv \max_{0 \leq e \leq 1} \{\pi(e)[\Phi(e) - \Phi(0) + e] + \Phi(0) - e\} \\ Q \; & = \max_{0 \leq e \leq 1} \left\{ \frac{\pi(e)}{1-\beta(1-\alpha)} \int_{\bar{w}}^\infty \Delta Y(w)F(dw) + 1 - e + z + \beta Q \right\} \end{aligned} \tag{31}$$

The right-hand side defines a mapping from Q into the reals. To guarantee that the problem is well behaved, we want to show that one such Q exists. This is not a trivial problem: Q affects $\bar{w}$ and $\Delta Y(w)$, so that the mapping is highly nonlinear. In any case, it is clear that Q, and therefore $\bar{w}$, are independent of e.

Let H be the mapping defined by the right-hand side of (31). Because $\pi(e)$ is increasing in e, we have that

$$\begin{aligned} HQ &\leq \tfrac{1}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) + 1 + z + \beta Q \\ &\leq \bar{H}Q \equiv \tfrac{1}{1-\beta(1-\alpha)} \int_{0}^{\infty} [w - T(w)] F(dw) + 1 + z + \beta Q. \end{aligned}$$

Therefore, if Q_1 is such that $Q_1 = \bar{H}Q_1$ (such a Q_1 is easy to compute directly), it follows that, for all $Q \geq Q_1$, $Q \geq \bar{H}Q$. Thus $\forall Q \geq Q_1$, $HQ \leq Q$. On the other hand,

$$\begin{aligned} HQ &\geq \left\{ \tfrac{\pi(0)}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) + 1 + z + \beta Q \right\} \\ &= 1 + z + \beta Q \equiv \underline{\mathrm{H}}Q. \end{aligned}$$

Then we have that, for all $Q \geq 0$, $\underline{\mathrm{H}}Q \leq HQ \leq \bar{H}Q$ and $\underline{\mathrm{H}}0 > 0$. Hence we have established that $H0 > 0$ and that there exists $Q_1 < \infty$ such that $HQ \leq Q$ for $Q \geq Q_1$.

Inasmuch as H is a continuous function of Q [this follows because $\bar{w}$ is continuous in Q, as is $\Delta Y(w)$], we establish that there exists a $\bar{Q}$ such that $H\bar{Q} = \bar{Q}$.

We next prove that $\bar{Q}$ is unique. To do so it suffices to show that the mapping H is monotone in Q. A sufficient condition is that

$$0 \leq \frac{\partial}{\partial \bar{w}} \left[\int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) \right] \frac{\partial \bar{w}}{\partial Q} + \beta < 1.$$

Still, $(\partial/\partial\bar{w}) \int_{\bar{w}}^{\infty} \Delta Y(w) F(dw)$ is (using the Leibniz rule) equal to $-[1-(\partial T/\partial w)(\bar{w})]$ $[1-F(\bar{w})]$. From the equation determining $\bar{w}$, we get that $[1-(\partial T/\partial w)(\bar{w})](\partial\bar{w}/\partial Q) = \beta(1-\beta)(1-\alpha)$. Because $-[1-F(\bar{w})]\beta(1-\beta)(1-\alpha) + \beta \in (0,1)$, however, H is increasing. Next we use (31) to characterize the optimal choice of e. It is clear that it satisfies

$$\pi'(e) \frac{1}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) = 1, \tag{32}$$

if the solution is interior. We assume that the distribution of w has sufficient mass in the tail to make search attractive – that is, we assume that the solution is interior. It is being claimed that it is possible to make assumptions about the deep parameters of the model, $F(w), \alpha, \beta, z, \pi(e)$, that will guarantee that the optimal choice of e is $e > 0$. We focus on this case only because the other is trivial.

From (31) it is clear that the optimal Q satisfies

$$\bar{Q} = (1-\beta)^{-1} \left[\frac{\pi(\bar{e})}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) + 1 - \bar{e} + z \right].$$

Using this equation in equation (30), we obtain another, more familiar characterization of the optimal reservation wage,

$$\begin{aligned} \bar{w} - T(\bar{w}) = \;& (1+z) - \beta(1-\alpha)\bar{e} + \tfrac{\beta(1-\alpha)\pi(\bar{e})}{1-\beta(1-\alpha)} \int_{\bar{w}}^{\infty} \{[w - T(w)] \\ & - [\bar{w} - T(\bar{w})]\} F(dw). \end{aligned} \tag{33}$$

Then equations (33) and (32) summarize the determination of the endogenous variables, e and $\bar{w}$.

b. Assume that $T(w) = t(w - a)$. To explore the effect of a change in a, we differentiate completely (33) and (32) with respect to a. We start with (33).

$$\begin{aligned}(1-t)\tfrac{\partial \bar{w}}{\partial a} + t = \;& -\beta(1-\alpha)\tfrac{\partial \bar{e}}{\partial a} + \tfrac{\beta(1-\alpha)}{1-\beta(1-\alpha)} \\ & \cdot \textstyle\int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) \pi'(\bar{e}) \tfrac{\partial e}{\partial a} \\ & - \tfrac{\beta(1-\alpha)\pi(\bar{e})}{1-\beta(1-\alpha)} (1-t)[1-F(\bar{w})] \tfrac{\partial \bar{w}}{\partial a}.\end{aligned}$$

Using equation (32) to eliminate $1/[1-\beta(1-\alpha)] \int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) \pi'(\bar{e}) = 1$, we get

$$(1-t)\left[1 + \frac{\beta(1-\alpha)\pi(\bar{e})[1-F(\bar{w})]}{1-\beta(1-\alpha)}\right] \frac{\partial \bar{w}}{\partial a} = -t.$$

Then $(\partial \bar{w}/\partial a) < 0$.
The intuition underlying this result is that an increase in a makes the income tax more progressive, as it increases the subsidy to low-income workers. Because taxes are paid (and the subsidy is received) only if the worker is employed, the increased attractiveness of low-income jobs is reflected by a reduction in the minimum wage at which an unemployed worker is willing to accept an offer. Notice that the term $(\partial e/\partial a)$ disappears in the above equation. This is just another consequence of the property that e does not affect the choice of the reservation wage.
We next explore the effect on e. From (32) we get

$$\begin{aligned}& \tfrac{\pi''(\bar{e})}{1-\beta(1-\alpha)} \textstyle\int_{\bar{w}}^{\infty} \Delta Y(w) F(dw) \tfrac{\partial e}{\partial a} \\ & \quad = \tfrac{\pi'(\bar{e})}{1-\beta(1-\alpha)} (1-t) \tfrac{\partial \bar{w}}{\partial a} [1-F(\bar{w})]\end{aligned}$$

or

$$\tfrac{\pi''(\bar{e})}{\pi'(\bar{e})^2} \tfrac{\partial e}{\partial a} = \tfrac{(1-t)[1-F(\bar{w})]}{1-\beta(1-\alpha)} \tfrac{\partial \bar{w}}{\partial a}.$$

Because $\pi''(e) < 0$, we have that $(\partial e/\partial a) > 0$, that is, effort is increased. Notice that the increase in e increases $\pi(\bar{e})$, and hence the probability of getting an acceptable offer $\pi(\bar{e})[1-F(\bar{w})]$ rises. To fix the notation, let $p = \pi(e)[1-F(\bar{w})]$. Then

$$\frac{\partial p}{\partial a} = [1-F(\bar{w})]\pi'(\bar{e}) \frac{\partial e}{\partial a} - F'(\bar{w})\pi(e) \frac{\partial \bar{w}}{\partial a},$$

and our results show that$(\partial p/\partial a) > 0$.

c. Next we analyze the effects of changing the marginal tax rate t. We follow exactly the same method of totally differentiating (33) and (32) to get, from (33),

$$\begin{aligned}& (1-t)\tfrac{\partial \bar{w}}{\partial t}\left\{1 + \tfrac{\beta(1-\alpha)\pi(\bar{e})[1-F(\bar{w})]}{1-\beta(1-\alpha)}\right\} \\ & \quad = \bar{w} - a - \tfrac{\beta(1-\alpha)\pi(\bar{e})}{1-\beta(1-\alpha)} \textstyle\int_{\bar{w}}^{\infty} (w-\bar{w}) F(dw).\end{aligned}$$

From (33), however, we got that

$$\begin{aligned}& \bar{w} - \tfrac{\beta(1-\alpha)\pi(\bar{e})}{1-\beta(1-\alpha)} \textstyle\int_{\bar{w}}^{\infty} (w-\bar{w}) F(dw) \\ & \quad = (1-t)^{-1}[(1+z) - a - \beta(1-\alpha)\bar{e}].\end{aligned}$$

Then

$$\text{sign } \frac{\partial \bar{w}}{\partial t} = \text{sign } [(1+z) - a - \beta(1-\alpha)\bar{e}].$$

From (32), after we substitute into the expression for $(\partial\bar{w}/\partial t)$, we get

$$\frac{\pi''(\bar{e})}{\pi'(\bar{e})}\frac{\partial e}{\partial t} = \frac{\pi'(\bar{e})}{[1-\beta(1-\alpha)+\beta(1-\alpha)\pi(\bar{e})[1-F(\bar{w})]} \cdot\left\{\bar{w} + \int_{\bar{w}}^{\infty}(w-\bar{w})F(dw)\right\}.$$

Therefore $(\partial e/\partial t) < 0$ unambiguously.
Notice that, in this case, an increase in t reduces the returns of being employed and therefore makes working less attractive. Consequently, it is optimal for the unemployed worker to reduce the level of effort, decreasing the probability of finding a job. On the other hand, it is possible for the reservation wage to decrease, that is, for some wage offers to be acceptable to the worker after the increase in the tax rate. Such a decrease becomes more likely as a grows larger. In this case, the increase in the marginal rate can actually increase payments to the worker when $w - a < 0$. This higher subsidy makes working more attractive, consequently reducing the reservation wage.

Exercise 5.12. *Search and nonhuman wealth*

An unemployed worker receives every period an offer to work forever at wage w, where w is drawn from the distribution $F(w)$. Offers are independently and identically distributed. Every agent has another source of income, which we denote ϵ_t, and that may be regarded as nonhuman wealth. In every period all agents get a realization of ϵ_t, which is independently and identically distributed over time, with distribution function $G(\epsilon)$. We also assume that w_t and ϵ_t are independent. The objective of a worker is to maximize

$$E\sum_{t=0}^{\infty}\beta^t y_t, \qquad 0<\beta<1,$$

where $y_t = w + \phi\epsilon_t$ if the worker has accepted a job that pays w, and $y_t = c + \epsilon_t$ if the worker remains unemployed. We assume that $0 < \phi < 1$ to reflect the fact that an employed worker has less time to engage in the collection of nonhuman wealth. Assume $1 > \text{prob}\{w \geq c + (1-\phi)\epsilon\} > 0$.
Analyze the worker's problem. Write down Bellman's equation and show that the reservation wage increases with the level of nonhuman wealth.

Solution

If the worker accepts a job that pays w, her total utility is given by

$$w + \theta\epsilon_t + E\sum_{j=1}^{\infty}\beta^j(w+\phi\epsilon_{t+j}) = w + \phi\epsilon + \frac{\beta}{1-\beta}(w+\phi E\epsilon).$$

Then let $v(w,\epsilon)$ be the optimal value of the objective function for an unemployed worker who has an offer w in hand and nonhuman wealth equal to ϵ. Then

$$v(w,\epsilon) = \max \left\{ w + \phi\epsilon + \tfrac{\beta}{1-\beta}(w + \phi E\epsilon) \atop c + \epsilon + \beta \int\int v(w',\epsilon')dF(w')dG(\epsilon') \right\}.$$

The second term in the bracketed expression does not depend on w. Therefore, for each ϵ, the optimal strategy is to choose a reservation wage. To see how the reservation wage $\bar{w}(\epsilon)$ varies with ϵ, write the indifference condition :

$$\bar{w}(\epsilon) + \phi\epsilon + \frac{\beta}{1-\beta}\left(\bar{w}(\epsilon) + \phi E(\epsilon)\right) = c + \epsilon + \beta Q,$$

where $Q \equiv \beta \int\int v(w',\epsilon')dF(dw')dG(\epsilon')$. Rearanging gives :

$$\frac{\bar{w}(\epsilon)}{1-\beta} = c + (1-\phi)\epsilon + \beta Q - \frac{\beta}{1-\beta}\phi E(\epsilon).$$

Since $0 < \phi < 1$, the above equation implies that $\bar{w}(\epsilon)$ is an increasing function of ϵ.

Exercise 5.13. *Search and asset accumulation*

A worker receives, when unemployed, an offer to work forever at wage w, where w is drawn from the distribution $F(w)$. Wage offers are identically and independently distributed over time. The worker maximizes

$$E\sum_{t=0}^{\infty}\beta^t u(c_t, l_t), \qquad 0 < \beta < 1,$$

where c_t is consumption and l_t is leisure. Assume R_t is i.i.d. with distribution $H(R)$. The budget constraint is given by

$$a_{t+1} \le R_t(a_t + w_t n_t - c_t)$$

and $l_t + n_t \le 1$ if the worker has a job that pays w_t. If the worker is unemployed, the budget constraint is $a_{t+1} \le R_t(a_t + z - c_t)$ and $l_t = 1$. Here z is unemployment compensation. It is assumed that $u(\cdot)$ is bounded and that a_t, the worker's asset position, cannot be negative. This corresponds to a no borrowing assumption. Write down Bellman's equation for this problem.

Solution

A natural choice for the state variable in this problem is the vector (w, a, R, s), where $s = E$ if the worker is employed and $s = U$ if the worker is unemployed.

We first analyze the problem faced by an employed worker. This problem is

$$v(w,a,R,E) = \max_{c,l,n,a'} \left\{ u(c,l) + \beta \int v(w,a',R',E)dH(R') \right\},$$

subject to $a' \leq R(a + wn - c)$, $l + n \leq 1$.
If the worker is unemployed, the value function is given by

$$v(w,a,R,U) = \max \Big\{ v(w,a,R,E), \max\Big[u(c,1) + \beta \int\int v(w',a',R',U)F(dw')dH(R')\Big]\Big\},$$

subject to $a' \leq R(a + z - c)$, where the first term in brackets reflects the value of accepting the job, whereas the second represents the value of remaining unemployed. In each case the asset position is chosen optimally. It is possible to argue that the optimal strategy is to set a reservation wage $\bar{w}(a,R)$ that depends on both the asset position and the rate of interest R.

Exercise 5.14. *Temporary unemployment compensation*

Each period an unemployed worker draws one, and only one, offer to work forever at wage w. Wages are i.i.d. draws from the c.d.f. F, where $F(0) = 0$ and $F(B) = 1$. The worker seeks to maximize $E\sum_{t=0}^{\infty} \beta^t y_t$, where y_t is the sum of the worker's wage and unemployment compensation, if any. The worker is entitled to unemployment compensation in the amount $\gamma > 0$ only during the *first* period that she is unemployed. After one period on unemployment compensation, the worker receives none.

a. Write the Bellman equations for this problem. Prove that the worker's optimal policy is a time-varying reservation wage strategy.

b. Show how the worker's reservation wage varies with the duration of unemployment.

c. Show how the worker's "hazard of leaving unemployment" (i.e., the probability of accepting a job offer) varies with the duration of unemployment.

Now assume that the worker is also entitled to unemployment compensation if she quits a job. As before, the worker receives unemployment compensation in the amount of γ during the first period of an unemployment spell, and zero during the remaining part of an unemployment spell. (To requalify for unemployment compensation, the worker must find a job and work for at least one period.)

The timing of events is as follows. At the very beginning of a period, a worker who was employed in the previous period must decide whether or not to quit. The decision is irreversible; that is, a quitter cannot return to an old job. If the worker quits, she draws a new wage offer as described previously, and if she accepts the offer she immediately starts earning that wage without suffering any period of unemployment.

d. Write the Bellman equations for this problem. [*Hint*: At the very beginning of a period, let $v^e(w)$ denote the value of a worker who was employed in the previous period with wage w (before any wage draw in the current period). Let $v_1^u(w')$ be the value of an unemployed worker who has drawn wage offer w' and who is entitled to unemployment compensation, if she rejects the offer. Similarly, let $v_+^u(w')$ be the value of an unemployed worker who has drawn wage offer w' but who is not eligible for unemployment compensation.]

e. Characterize the three reservation wages, $\bar{w}^e$, $\bar{w}_1^u$, and $\bar{w}_+^u$, associated with the value functions in part d. How are they related to γ? (*Hint*: Two of the reservation wages are straightforward to characterize, while the remaining one depends on the actual parameterization of the model.)

Solution

a. Let $v_u^1(w)$ ($v_+^u(w)$) be the value function of an unemployed worker with wage w in hand in the first (after the first) period of unemployment and who behaves optimally. The Bellman equation are :

$$\begin{aligned} v_1^u(w) &= \max_{\{A,R\}} \left\{ \frac{w}{1-\beta}, \gamma + \beta \int_0^B v^+(w')dF(w') \right\} \\ v_+^u(w) &= \max_{\{A,R\}} \left\{ \frac{w}{1-\beta}, \beta \int_0^B v^+(w')dF(w') \right\}. \end{aligned}$$

In each of the two periods the problem is a standard one, leading to a reservation wage policy. If the unemployed is in her fist period of unemployment then the optimal policy is accept for $w \geq w^1$ and reject otherwise. The associated value function is $v_1^u(w) = \frac{w}{1-\beta}$ for $w \geq w^1$ and $v_1^u(w) = \frac{w^1}{1-\beta} = \gamma + \beta \int_0^B v_1^u(w')dF(w')$ for $w < w^1$. After one (or more) period(s) of unemployment the optimal policy is to accept when $w \geq w^+$ and to reject otherwise.

b. To show that $w^1 > w^+$, just write the two indifference conditions satisfied by the two reservation wages :

$$\begin{aligned} \tfrac{w^1}{1-\beta} &= \gamma + \beta \textstyle\int_0^B v_+^u(w')dF(w') \\ \tfrac{w^+}{1-\beta} &= \beta \textstyle\int_0^B v_+^u(w')dF(w'). \end{aligned}$$

Clearly, $w^1 > w^+$. Note that $w^1 - w^+ = (1-\beta)\gamma$. This equality has the following interpretation. Suppose that an unemployed worker in the first period of unemployment receive an offer w. If he accepts it, her payoff is :

$$\frac{w}{1-\beta}.$$

If, on the other hand, she rejects it, then her payoff is made of two terms. The first term is the unemployment compensation, γ. The second term is the option value of waiting, which is equal to $\frac{w^+}{1-\beta}$. Thus, the worker accepts whenever :

$$w \geq \gamma(1-\beta) + w^+.$$

in term of "average payoff" per period, the worker accepts whenever the wage exceed the reservation wage w^+ plus the annuity value of receiving unemployment compensation today.

c. The workers probability of finding a job is determined by $P[w > w^i]$, $i = +, 1$. Since $w^1 > w^+$, the probability of accepting a job is higher after one period of unemployment: $P[w > w^+] \geq P[w > w^1]$.

d. $v_1^u(w)$ and v_+^u are defined as in question a. Let $v^e(w)$ be the value of an employed worker with wage w in hand and who behaves optimally. The three value functions are solution of the following system of Bellman equations :

$$v^e(w) = \max_{\text{stay,quit}} \left\{ w + \beta v^e(w), \int_0^B v_1^u(w')dF(w') \right\} \tag{34}$$

$$v_1^u(w) = \max_{\text{accept,reject}} \left\{ w + \beta v^e(w), \gamma + \beta \int_0^B v_+^u(w')dF(w') \right\} \tag{35}$$

$$v_+^u(w) = \max_{\text{accept,reject}} \left\{ w + \beta v^e(w), \beta \int_0^B v_+^u(w')dF(w') \right\}. \tag{36}$$

e. To simplify notations, define first $Q_1 \equiv \int_0^B v_1^u(w')dF(w')$ and $Q_+ \equiv \int_0^B v_+^u(w')dF(w')$. The characterization goes in several steps.

Step 1 : Characterizing $v^e(w)$

From equation (34) it is clear that if an employed worker decides to stay in a given period, he will decide to stay in all the subsequent periods. Thus we can rewrite equation (34) as :

$$v^e(w) = \max_{\text{stay,quit}} \left\{ \frac{w}{1-\beta}, Q_1 \right\}.$$

Furthermore, the above expression shows that the optimal policy of an employed worker is of the reservation wage form. Specificaly, there exists w_e such that for all $w \leq w_e$ the worker quits her job and $v^e(w) = Q_1$. For all $w > w_e$ the worker stays at her job and $v^e(w) = \frac{w}{1-\beta}$. Lastly, w_e solves the following indifference condition :

$$\frac{w_e}{1-\beta} = Q_1.$$

Step 2 : Unemployed workers have reservation wage policies

Since $v^e(w)$ is increasing, it follows from the Bellman equations (35) and (36) that unemployed workers have reservation wage policies. Let w^1 and w^+ be the corresponding reservation wages.

Step 3 : $Q_1 \geq Q_+$

Equations (35) and (36) imply that $v_1^u(w) \geq v_+^u(w)$ for all w. In word, unemployed workers are better off in the first period of unemployment because they receive the benefit γ. Integrating with respect to $dF(w')$ implies that $Q_1 \geq Q_+$.

Step 4 : $w^+ = 0$

This is an intuitive fact. After the first period of unemployment a worker does not receive any benefit. Accepting an offer and quitting is as least as good as rejecting an offer and drawing again next period. Let's prove it formally. Assume $w^+ > 0$. Then $w^+ + \beta v_+^u(w^+) = \beta Q_+ < Q_1 = w_e + \beta v^e(w^e)$. Since $w + v^e(w)$ is weakly increasing this implies that $w^+ < w^e$. Thus $v^e(w^+) = Q_1$. Thus $w^+ + \beta Q_1 = \beta Q_+$, implying that $w^+ = \beta(Q_+ - Q_1) < 0$. A contradiction.

Step 5 : $\gamma > 0 \Rightarrow w^1 > 0$

This is also an intuitive result. Since an unemployed workers receives benefits in its first period of unemployment and, she surely refuses to work when the wage offer is small enough. Formally, assume that $w_1 = 0$. Then $v_1^u(w) = v_+^u(w)$ for all w. This implies that $Q_1 = Q_+$. Also, since $w_1 = 0$, accepting wage offer 0 is at least as good as rejecting it. Thus :

$$0 + \beta v^e(0) = \beta Q_1 \geq \gamma + \beta Q_+ = \gamma + \beta Q_1.$$

When we use the fact that $v^e(0) = Q_1$. The above implies $\gamma < 0$. A contradiction.

Step 6 : $w_1 \leq w_e$ *and* $w_1 = \gamma - \beta(Q_1 - Q_+)$

The Bellman equation (35) implies that $v_1^u(w) \geq \gamma + \beta Q_+$ for all w. Integrating with respect to $dF(w')$ gives $Q_1 \geq \gamma + \beta Q_+$. From the indifference conditions defining w^e and w^1, this is equivalent to :

$$w^e + \beta v^e(w^e) \geq w^1 + \beta v^e(w^1).$$

Since v^e is weakly increasing, it implies that $w^e \geq w^1$. Thus $v^e(w^1) = Q_1$. Using this equality to rewrite the indifference condition defining w^1 gives :

$$w^1 + \beta Q_1 = \gamma + \beta Q_+ \Rightarrow w^1 = \gamma + \beta(Q_+ - Q_1).$$

The above manipulations show that $0 = w^+ \leq w^1 \leq w^e$ and $w^1 = \gamma + \beta(Q_+ - Q_1)$.

Note that w_1 is strictly less than γ. This reflect the fact that, when an agent reject, she receives unemployment compensation this period but also loose the right to receive it next period. On the other hand,if she accepts, she keeps the right to receive unemployment compensation next period.
Lastly, we cannot tell whether or not w^e is smaller or greater than gamma.

Step 6 : Dependence on γ

First note that the value functions are weakly increasing in γ. To see why this is the case consider the optimization problem when the compensation is $\gamma' = \gamma + \Delta\gamma > \gamma$. A posible decision rule for the agents is to use the same reservation wage policy as when the compensation is γ. Payoffs are the same as before except for an additional $\Delta\gamma$ in the first period of unemployment. Therefore, the value of using this decision rule has increased. Now, under the *optimal* decision rule, the value is necessarily even larger.

Since $v_1^u(w,\gamma)$ is weakly increasing in γ, $Q_1 = \int_0^B v_1^u(w',\gamma)dF(w')$ is also weakly increasing in γ. Thus $w_e = (1-\beta)Q_1$ is weakly increasing in γ.

Exercise 5.15. *Seasons, I*

An unemployed worker seeks to maximize $E\sum_{t=0}^{\infty}\beta^t y_t$, where $\beta \in (0,1)$, y_t is her income at time t, and E is the mathematical expectation operator. The person's income consists of one of two parts: unemployment compensation of c that she receives each period she remains unemployed, or a fixed wage w that the worker receives if employed. Once employed, the worker is employed forever with no chance of being fired. Every odd period (i.e., $t = 1, 3, 5, \ldots$) the worker receives one offer to work forever at a wage drawn from the c.d.f. $F(W) = \text{prob}(w \leq W)$. Assume that $F(0) = 0$ and $F(B) = 1$ for some $B > 0$. Successive draws from F are independent. Every even period (i.e., $t = 0, 2, 4, \ldots$), the unemployed worker receives two offers to work forever at a wage drawn from F. Each of the two offers is drawn independently from F.

a. Formulate the Bellman equations for the unemployed person's problem.

b. Describe the form of the worker's optimal policy.

Solution

a. The Bellman equations for the even periods $V^e(w)$ and for the odd periods $V^o(w)$ are:

$$\begin{aligned}
V^o(w) &= \max_{\{A,R\}}\left\{\frac{w}{1-\beta}, c + \beta\int_0^B V^e(w')dF^2(w')\right\} \\
V^e(w) &= \max_{\{A,R\}}\left\{\frac{w}{1-\beta}, c + \beta\int_0^B V^o(w')dF(w')\right\}.
\end{aligned}$$

b. The workers optimal policy will be a reservation wage in each period below which the worker refuses the best offer outstanding and above which she accepts. For the odd periods, this reservation wage obeys the equation

$$\begin{aligned}
\frac{\bar{w}^o}{1-\beta} &= c+\beta\int_0^{\bar{w}^e}\frac{\bar{w}^e}{1-\beta}dF^2(w')+\beta\int_{\bar{w}^e}^B\frac{w'}{1-\beta}dF^2(w') \\
c &= \int_0^{\bar{w}^e}\frac{\bar{w}^o-\beta\bar{w}^e}{1-\beta}dF^2(w')+\int_{\bar{w}^e}^B\frac{(\bar{w}^o-\beta w')}{1-\beta}dF^2(w') \\
c(1-\beta) &= \left(\bar{w}^o-\beta\bar{w}^e\right)F^2(\bar{w}^e)+\bar{w}^o\left(1-F^2(\bar{w}^e)\right)-\beta\int_{\bar{w}^e}^B w'dF^2(w'),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\bar{w}^e}{1-\beta} &= c+\beta\int_0^{\bar{w}^o}\frac{\bar{w}^o}{1-\beta}dF(w')+\beta\int_{\bar{w}^o}^B\frac{w'}{1-\beta}dF(w') \\
c &= \int_0^{\bar{w}^o}\frac{\bar{w}^e-\beta\bar{w}^o}{1-\beta}dF(w')+\int_{\bar{w}^o}^B\frac{\bar{w}^e-\beta w'}{1-\beta}dF(w') \\
c(1-\beta) &= \left(\bar{w}^e-\beta\bar{w}^o\right)F(\bar{w}^o)+\bar{w}^e\left(1-F^2(\bar{w}^o)\right)-\beta\int_{\bar{w}^o}^B w'dF(w').
\end{aligned}$$

Equating the two expressions for $c(1-\beta)$:

$$\begin{aligned}
\bar{w}^o-\beta\bar{w}^eF^2(\bar{w}^e)-\beta\int_{\bar{w}^e}^B w'dF^2(w') &= \bar{w}^e-\beta\bar{w}^oF(\bar{w}^o)-\beta\int_{\bar{w}^o}^B w'dF(w') \\
\bar{w}^o+\beta\left[\bar{w}^oF(\bar{w}^o)+\int_{\bar{w}^o}^B w'dF(w')\right] &= \bar{w}^e+\beta\left[\bar{w}^eF^2(\bar{w}^e)+\int_{\bar{w}^e}^B w'dF^2(w')\right].
\end{aligned}$$

For a given $\bar{w}$, we know that $\bar{w}F^2(\bar{w})+\int_{\bar{w}}^B w'dF^2(w')\geq\bar{w}F(\bar{w})+\int_{\bar{w}}^B w'dF(w')$. Furthermore, using Leibnitz rule, we know that $\bar{w}F(\bar{w})+\int_{\bar{w}}^B w'dF(w')$ and $\bar{w}F^2(\bar{w})+\int_{\bar{w}}^B w'dF^2(w')$ are increasing in $\bar{w}$. Using these two facts, the above equality cannot hold for $\bar{w}^o<\bar{w}^e$, because both terms on the left hand side would be less than the corresponding terms on the right hand side. We conclude that $\bar{w}^o\geq\bar{w}^e$. The intuition is that in odd periods, the unemployed worker's outside option (reject and two draws next period) is better than his outside option in even periods (reject and sample once next period). That makes him want a higher reservation wage in odd periods.

Exercise 5.16. *Seasons, II*

Consider the following problem confronting an unemployed worker. The worker wants to maximize

$$E_0\sum_0^\infty\beta^ty_t,\quad\beta\in(0,1),$$

where $y_t=w_t$ in periods in which the worker is employed and $y_t=c$ in periods in which the worker is unemployed, where w_t is a wage rate and c is a constant level of unemployment compensation. At the start of each period, an unemployed worker receives one and only one offer to work at a wage w drawn from a c.d.f. $F(W)$, where $F(0)=0, F(B)=1$ for some $B>0$. Successive draws from F

are identically and independently distributed. There is no recall of past offers. Only unemployed workers receive wage offers. The wage is fixed as long as the worker remains in the job. The only way a worker can leave a job is if she is fired. At the *beginning* of each odd period ($t = 1, 3, \ldots$), a previously employed worker faces the probability of $\pi \in (0,1)$ of being fired. If a worker is fired, she immediately receives a new draw of an offer to work at wage w. At each even period ($t = 0, 2, \ldots$), there is no chance of being fired.

a. Formulate a Bellman equation for the worker's problem.

b. Describe the form of the worker's optimal policy.

Solution

a. Let $v_U^e(w)$, $(v_U^o(w))$ be the value an uemployed worker who has just received an offer w at the start of an even (odd) period and proceeds optimally. Similarly, let $v_E^e(w)$, $(v_E^o(w))$ be the value an employed with wage w the beginning of an even (odd) period. The Bellman equation for v_U^e is :

(37)
$$v_U^e(w) = \max\left\{w + \beta\left[\pi\int_0^B v_U^o(w')dF(w') + (1-\pi)v_E^o(w)\right], c + \beta\int_0^B v_U^o(w')dF(w')\right\}.$$

Similarly, the Bellman equation for v_U^o is :

$$v_U^o(w) = \max\left\{w + v_E^e(w), c + \beta\int_0^B v_U^e(w')dF(w')\right\}. \tag{38}$$

Bellman equations for employed workers are :

$$v_E^o(w) = w + \beta v_E^e(w) \tag{39}$$

$$v_E^e(w) = w + \beta\pi\int_0^B v_U^o(w')dF(w') + (1-\pi)v_E^o(w). \tag{40}$$

There is no "max" in the two Bellman equations because it was assumed that the only way an employed worker can leave a job is by being fired. These 4 Bellman equations fully describe the worker's dynamic choice problem.

b. First, solve for $v_E^o(w)$ and $v_E^e(w)$ in terms of the optimum value functions. This produces:

$$v_E^e(w) = \frac{w\left(1 + (1-\pi)\beta\right) + \beta\pi\int v_U^o(w')dF(w')}{1-(1-\pi)\beta^2}$$

$$v_E^o(w) = \frac{w\left(1+\beta\right) + \beta^2\pi\int v^o(w')dF(w')}{1-(1-\pi)\beta^2}.$$

Replacing those expressions in (37) and (38) shows that a reservation wage policy is optimal in this setting. Let the w^e (w^o) be the reservation wage in even (odd) periods. The indifference conditions are :

$$w^e + \beta\left[\pi \int_0^B v_U^o(w')dF(w') + (1-\pi)v_E^o(w^e)\right] = c + \beta \int_0^B v_U^o(w')dF(w'),$$

which implies that:

$$w^e \frac{1+\beta(1-\pi)}{1-(1-\pi)\beta^2} + \frac{\beta^2\pi}{1-(1-\pi)\beta^2} Q^o = c + \beta Q^o,$$

where $Q^o = \equiv \int_0^B v_U^o(w')dF(w')$. Some simple algebra yields:

$$w^e = \frac{1-(1-\pi)\beta^2}{1+\beta(1-\pi)}c + \frac{\beta\left[1-(1-\pi)\beta^2\right]Q^o - \beta^2\pi Q^o}{1+\beta(1-\pi)} \tag{41}$$

$$= \frac{1-(1-\pi)\beta^2}{1+\beta(1-\pi)}c + \frac{\beta\left[1-(\pi+\beta(1-\pi))\beta^2\right]}{1+\beta(1-\pi)}Q^o. \tag{42}$$

Similarly, we know from (38) that, at the reservation wage w^o:

$$w^o + \beta\left[v_E^e(w^o)\right] = c + \beta Q^e, \tag{43}$$

which in turn implies that:

$$w^o \frac{(1-\beta)}{1-(1-\pi)\beta^2} + \frac{\beta^2\pi}{1-(1-\pi)\beta^2} Q^o = c + \beta Q^e.$$

Some simple algebra yields:

$$w^o = \frac{1-(1-\pi)\beta^2}{1-\beta}c + \frac{\beta\left[1-(1-\pi)\beta^2\right]Q^e - \beta^2\pi Q^o}{1-\beta}. \tag{44}$$

Now, we can compare the reservation wages in even and odd periods by comparing eqs. (41) and (44). It is easy to verify that $w^o \geq w^e$ whenever $Q^e \geq Q^o$, since $\beta \in (0,1)$ and $\pi \in (0,1)$. This means that, if the optimum value of the unemployed (or, equivalently, just fired) worker is higher at the start of an even period than at the start of an odd period, then the worker sets a higher reservation wage in the odd period, because he is quite willing to wait another period while being unemployed (in order to receive Q^e). Also note that the reverse statement is not necessarily true.

Exercise 5.17. *Gittins indices for beginners*

At the end of each period, a worker can switch between two jobs, A and B, to begin the following period at a wage that will be drawn at the beginning of next period from a wage distribution specific to job A or B, and to the worker's history of past wage draws from jobs of either type A or type B. The worker must decide to stay or leave a job at the end of a period after his wage for this period on his current job has been received, but before knowing what his wage would be next period in either job. The wage at either job is described by a job-specific

n-state Markov chain. Each period the worker works at either job A or job B. At the end of the period, before observing next period's wage on either job, he chooses which job to go to next period. We use lowercase letters $(i, j = 1, \ldots, n)$ to denote states for job A, and uppercase letters $(I, J = 1, \ldots n)$ for job B. There is no option of being unemployed.

Let $w_a(i)$ be the wage on job A when state i occurs and $w_b(I)$ be the wage on job B when state I occurs. Let $A = [A_{ij}]$ be the matrix of one-step transition probabilities between the states on job A, and let $B = [B_{ij}]$ be the matrix for job B. If the worker leaves a job and later decides to returns to it, he draws the wage for his first new period on the job from the conditional distribution determined by his last wage working at that job.

The worker's objective is to maximize the expected discounted value of his lifetime earnings, $E_0 \sum_{t=0}^{\infty} \beta^t y_t$, where $\beta \in (0,1)$ is the discount factor, and where y_t is his wage from whichever job he is working at in period t.

a. Consider a worker who has worked at both jobs before. Suppose that $w_a(i)$ was the last wage the worker receives on job A and $w_b(I)$ the last wage on job B. Write the Bellman equation for the worker.

b. Suppose that the worker is just entering the labor force. The first time he works at job A, the probability distribution for his initial wage is $\pi_a = (\pi_{a1}, \ldots, \pi_{an})$. Similarly, the probability distribution for his initial wage on job B is $\pi_b = (\pi_{b1}, \ldots, \pi_{bn})$ Formulate the decision problem for a new worker, who must decide which job to take initially. [*Hint*: Let $v_a(i)$ be the expected discounted present value of lifetime earnings for a worker who was last in state i on job A and has never worked on job B; define $v_b(I)$ symmetrically.]

Solution

a. First we consider a worker who has worked at both jobs before. Suppose that $w_a(i)$ was the last wage the worker receives at job A and $w_b(I)$ was the last wage he received at job B.
Let $v(i, I)$ be the optimum value, starting from next period, of a worker currently active in job A at wage $w_a(i)$ who has also worked at job B (at some point in the past) at a wage $w_B(I)$. Again, this worker is at the end of the current period and has to decide where to go in the next period **before** having observed next period's wage on either job. Similarly, let $v(I, i)$ be the optimum value, starting from next period, of a worker currently active in job B at wage $w_b(I)$ who has also worked at job A (at some point in the past) at a wage $w_A(i)$.
The Bellman equation for the first worker is given by:

$$v(i, I) = \max_{A,B} \left\{ \sum_{j=1}^{n} A_{ij} \left[w_A(j) + \beta v(j, I) \right], \sum_{J=1}^{n} B_{IJ} \left[w_B(J) + \beta v(J, i) \right] \right\}, \tag{45}$$

while the Bellman equation of the second worker is given by:

$$v(I,i) = \max_{A,B} \left\{ \sum_{j=1}^{n} A_{ij} \left[w_A(j) + \beta v(j,I) \right], \sum_{J=1}^{n} B_{IJ} \left[w_B(J) + \beta v(J,i) \right] \right\}. \tag{46}$$

Notice how $v(i,I) = v(I,i)$ by comparing the r.h.s of eq. (45) and eq. (46). This implies we can let $v(i,I)$ denote the optimum value of a worker whose last wage at job A was $w_a(i)$ and at job B was $w_b(I)$. (Let's agree on making the first argument of the value function the last wage at A).

$$v(i,I) = \max_{A,B} \left\{ \sum_{j=1}^{n} A_{ij} \left[w_A(j) + \beta v(j,I) \right], \sum_{J=1}^{n} B_{IJ} \left[w_B(J) + \beta v(i,J) \right] \right\}. \tag{47}$$

b. Next, we turn to consider the problem facing a worker who is just about to enter the labor force. Working backwards, we first examine the case of a worker who has only worked on one job. Let $v_a(i)$ denote the optimum value of a worker at job A, making a wage $w_a(i)$, who has never worked on job B before and let $v_b(I)$ denote the same value for a worker earning a wage $w_b(I)$ at B, who has never worked job A before. Then we know that the Bellman equation of a worker who has only worked at A is:

$$v_a(i) = \max_{A,B} \left\{ \sum_{j=1}^{n} A_{ij} \left[w_a(j) + \beta v_a(j) \right], \sum_{J=1}^{n} \pi_b(J) \left[w_b(J) + \beta v(i,J) \right] \right\},$$

while the Bellman equationof a worker who has only worked at B is given by:

$$v_b(I) = \max_{A,B} \left\{ \sum_{j=1}^{n} \pi_a(j) \left[w_a(j) + \beta v(j,J) \right], \sum_{J=1}^{n} B_{IJ} \left[w_b(J) + \beta v_b(J) \right] \right\},$$

Finally, consider the problem facing a worker who is about to enter the labor force; naturally, she starts at the job that yields the highest expected lifetime utility:

$$Q = \max_{A,B} \left\{ \sum_{i=1}^{n} \pi_a(i) \left[w_a(i) + \beta v_a(i) \right], \sum_{I=1}^{n} \pi_a(I) \left[w_b(I) + \beta v_b(I) \right] \right\}.$$

Now we have exhaustively described the worker's problem, proceeding **backwards**, which is the only way to solve this type of problem.

Exercise 5.18. *Jovanovic (1979b)*

An employed worker in the tth period of tenure on the current job receives a wage $w_t = x_t(1 - \phi_t - s_t)$ where x_t is job-specific human capital, $\phi_t \in (0,1)$ is the fraction of time that the worker spends investing in job-specific human capital, and $s_t \in (0,1)$ is the fraction of time that the worker spends searching for a new job offer. If the worker devotes s_t to searching at t, then with probability $\pi(s_t) \in (0,1)$ at the beginning of $t+1$ the worker receives a new job offer to begin working at new job-specific capital level μ' drawn from the c. d. f. $F(\cdot)$. That is, searching for a new job offer promises the prospect of instantaneously reinitializing job-specific human capital at μ'. Assume that $\pi'(s) > 0, \pi''(s) < 0$. While on a given job, job-specific human capital evolves according to

$$x_{t+1} = G(x_t, \phi_t) = g(x_t \phi_t) - \delta x_t,$$

where $g'(\cdot) > 0, g''(\cdot) < 0$, $\delta \in (0,1)$ is a depreciation rate, and $x_0 = \mu$ where t is tenure on the job, and μ is the value of the "match" parameter drawn at the start of the current job. The worker is risk neutral and seeks to maximize $E_0 \sum_{\tau=0}^{\infty} \beta^\tau y_\tau$, where y_τ is his wage in period τ.

a. Formulate the worker's Bellman equation.

b. Describe the worker's decision rule for deciding whether to accept an offer μ' at the beginning of next period.

c. Assume that $g(x\phi) = A(x\phi)^\alpha$ for $A > 0, \alpha \in (0,1)$. Assume that $\pi(s) = s^{.5}$. Assume that F is a discrete n-valued distribution with probabilities f_i; for example, let $f_i = n^{-1}$. Write a Matlab program to solve the Bellman equation. Compute the optimal policies for ϕ, s and display them.

Solution

a. Let $v(x)$ be the optimum value at the start of the current period of an employed worker who has accumulated a total amount x of job-specific capital and who proceeds optimally. We know the worker will accept the new draw μ' at the start of next period whenever μ' exceeds next period's capital on the old job x'. This means her Bellman equation is given by:

$$v(x) = \max_{\phi,s} x(1-\phi-s) + \beta \left[((1-\pi(s))v(x')) + \pi(s) \int \max(v(\mu'), v(x'))dF(\mu') \right]$$

$$v(x) = \max_{\phi,s} x(1-\phi-s) + \beta \left[((1-\pi(s))v(x')) + \pi(s) \left[\int_{x'} v(\mu')dF(\mu') + F(x')v(x') \right] \right]$$

,
where $x' = G(x, \phi) = g(x\phi) - \delta x$

b. The question is answered in part a.

c. The matlab code is in zia.stanford.edu/public/sarg/webdocs/teaching/econ210/ in files `jova.m` and `readjova.txt`