Problem Set 5
Answer Key

December 16, 2016

Problem 1. McCall meets DMP
Consider the standard DMP model discussed in the class with the following modification: When a worker and a firm meet they draw a match specific productivity, z from a continuous distribution $G(z)$, suppose that z has a Pareto distribution given by

$$G(z) = 1 - \left(\frac{z}{z_0}\right)^{-\zeta}$$

This productivity lasts for the duration of the job - until separation. The rest is identical to the model in class - separation rate is independent of the job productivity. Suppose that for each level of productivity that is accepted by firms and workers, there is an equilibrium wage $w(z)$. Assume that this wage is determined via Nash bargaining between the firm and the worker.

a. Write the Bellman equations for the workers and firms as a function of the wage function $w(z)$.

Solution. Let $q(\theta_t) = \frac{m(u_t,v_t)}{v_t}$ be the probability of filling a vacancy and $\theta_t q(\theta_t) = \frac{m(u_t,v_t)}{u_t}$ be the probability of finding a job.

The value of unemployment for the worker:

$$\rho U = b + \theta q(\theta) \left[\int_{z^*} \left(V(w(z)) - U\right) dG(z)\right] \tag{1}$$

where z^* is the lowest level of productivity for which a match is formed. That is total surplus of the match $J(w) + V(w) - U - W$ is positive.

The value of working at a job at the wage rate $w(z)$:

$$\rho V(w(z)) = w(z) + \lambda[U - V(w(z))] \tag{2}$$

The value of an unfilled vacancy for the firm:

$$\rho W = -\kappa + q(\theta) \left[\int_{z^*} \left(J(w(z)) - W\right) dG(z)\right] \tag{3}$$

The value of having a worker at the wage rate $w(z)$ for the firm:

$$\rho J(w(z)) = z - w(z) + \lambda[W - J(w(z))] \tag{4}$$

b. Characterize the value of reservation wage.

Solution. When a productivity arrives at t then the worker and the firm, knowing that they will bargain in the future, will only accept a productivity draw if under bargaining the total surplus from this is positive. Thus, at z^* , we must have that

$$J\left(w\left(z^*\right)\right)+V\left(w\left(z^*\right)\right)-W-U=0$$

If we subtract, ρU from (2) and ρW from (4), then we have that

$$(\rho+\lambda)\left[J\left(w\left(z^*\right)\right)+V\left(w\left(z^*\right)\right)-W-U\right]=z-\rho U-\rho W$$

Therefore, z^* must satisfy

$$z^*=\rho U+\rho W$$

Now, in order to calculate the reservation wage, i.e., the wage associated with lowest productivity that is accepted by both firm and the worker, we need to fully specify the value of wages that result from the bargaining process. In order to do this, recall that wages must solve

$$\max_w (V(w)-U)^\gamma (J(w)-W)^{1-\gamma}$$

As before, $-J'\left(w\right)=V'\left(w\right)=\frac{1}{\rho+\lambda}$. As a result, we have

$$\frac{\gamma}{V\left(w\right)-U}=\frac{1-\gamma}{J\left(w\right)-W}$$

From above,

$$\begin{aligned}
V\left(w\right)-U &= \frac{w-\rho U}{\rho+\lambda} \\
J\left(w\right)-W &= \frac{z-w-\rho W}{\rho+\lambda}
\end{aligned}$$

Therefore,

$$w\left(z\right)=\gamma\left(z-\rho W\right)+\left(1-\gamma\right)\rho U$$

In particular at z^*,

$$\begin{aligned}
w\left(z^*\right) &= \gamma\left(z^*-\rho W-\rho U\right)+\rho U \\
&= \rho U
\end{aligned}$$

When $W=0$, then

$$w\left(z^*\right)=\rho U=z^*$$

Interestingly, since the worker always get a constant share of the surplus, the above implies that the wage above is in fact a reservation wage and therefore, must satisfy

$$w\left(z^*\right)=z^*=b+\frac{\theta q\left(\theta\right)}{\rho+\lambda}\gamma\int_{z^*}^{\infty}\left(z-z^*\right)dG\left(z\right)$$

Since z has a pareto distribution,

$$\begin{aligned}\int_{z^*}^{\infty}(z-z^*)\,dG(z) &= \int_{z^*}^{\infty}(z-z^*)\,\zeta z_0^{\zeta} z^{-1-\zeta}dz\\ &= \int_{z^*}^{\infty}\zeta z_0^{\zeta} z^{-\zeta}dz - z^*\int_{z^*}^{\infty}\zeta z_0^{\zeta} z^{-1-\zeta}dz\\ &= \frac{\zeta}{\zeta-1}z_0^{\zeta}(z^*)^{1-\zeta} - z^* z_0^{\zeta}(z^*)^{-\zeta}\\ &= \frac{1}{\zeta-1}z_0^{\zeta}(z^*)^{1-\zeta}\end{aligned}$$

Therefore

$$z^* = b + \frac{\theta q(\theta)}{\rho+\lambda}\gamma\frac{1}{\zeta-1}z_0^{\zeta}(z^*)^{1-\zeta}$$

This equation relates $z^* = \rho U$ to market tightness.

c. Derive the steady-state equilibrium conditions.

Solution. In the steady state in-flow equals out-flow of workers: unemployed now is equal to the unemployed people who could not find a job and employed people who lost their jobs. That is

$$\begin{aligned}u &= (1-u)\lambda + u(1-\theta q(\theta))\\ u &= \frac{\lambda}{\lambda+\theta q(\theta)}\end{aligned}$$

Free entry at the firms side implies that firms will be entering to the market until the value of vacancy W is zero. Then we have

$$\begin{aligned}\kappa &= q(\theta)\int_{\rho U}^{\infty}\frac{(1-\gamma)(z-\rho U)}{\rho+\lambda}dG(z)\\ \rho U &= b + \theta q(\theta)\int_{\rho U}^{\infty}\frac{\gamma(z-\rho U)}{\rho+\lambda}dG(z)\end{aligned}$$

After some manipulations on the above, we have

$$\frac{\rho U - b}{\kappa} = \theta\frac{\gamma}{1-\gamma} \Rightarrow z^* = \rho U = b + \kappa\theta\frac{\gamma}{1-\gamma} \tag{5}$$

The above is another equation in terms of θ and z^*.

d. How does the wage distribution depend on the unemployment benefits?

Solution. From the free entry condition above, we have that

$$\kappa = q(\theta)\frac{(1-\gamma)}{\rho+\lambda}\frac{1}{\zeta-1}z_0^{\zeta}(z^*)^{1-\zeta}$$

This is an equation that leads to a decreasing relationship between z^* and θ. Moreover, it implies that $\theta = q^{-1}\left(\kappa\frac{(\rho+\lambda)(\zeta-1)}{1-\gamma}\left(z^*\right)^{\zeta-1} z_0^{\zeta}\right)$. Recall that for a pareto distribution to have a well-defined mean, $\zeta > 1$ and since q is decreasing, we have that θ decreases in z^*. Replacing in (5), we have

$$z^* - q^{-1}\left(\kappa\frac{(\rho+\lambda)(\zeta-1)}{1-\gamma}\left(z^*\right)^{\zeta-1} z_0^{\zeta}\right)\kappa\frac{\gamma}{1-\gamma} = b$$

The LHS of the above is increasing in z^*. Hence, an increase in b must lead to an increase in z^*. This means that an increase in b shifts upward the distribution of wages. Since a pareto distribution keeps its shape when its lower bound changes, nothing else happens to the distribution.

Problem 2.
Solve problems 6.7, 6.8, 6.10, 6.19 in Ljungqvist and Sargent.